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CONTENTS

Some Factors Affecting Large Transport Aeroplanes with Turboprop Engines (Wright Brothers Lecture).....A. E. RUSSELL 67

Application of Simultaneous Equation Machines to Aircraft Structure and Flutter Problems.....P. A. DENNIS AND D. G. DILL 107

A Note on the Approximate Plane Motion During the Burning Period of Rocket-Propelled Missiles Launched at Small Angles of Yaw from Aircraft..RAY E. BOLZ 114

Seventy-Fifth Anniversary of Ludwig Prandtl..... 121

Readers' Forum: "Note on Rotational Gas Flow," by James E. Broadwell; "The Mechanism of Unstable Combustion," by W. C. Randels; "Detonation and Static Electricity in the Tropical Atmosphere," by G. Broersma; "Second-Order Two-Dimensional Supersonic Airfoil Theory," by Bruno W. Augenstein; "On Simple Plan Forms in Supersonic Flow," by John W. Miles.....123–127

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JOURNAL OF THE AERONAUTICAL SCIENCES

VOLUME 17 FEBRUARY, 1950 NUMBER 2

Some Factors Affecting Large Transport Aeroplanes with Turboprop Engines

The Thirteenth Wright Brothers Lecture*

A. E. RUSSELL†

The Bristol Aeroplane Company, Ltd., England

Foreword

To deliver a lecture in commemoration of the Wright Brothers before the Institute of Aeronautical Sciences is a great honour of which I am deeply aware. This honour one feels is due not solely to the technical situation but to some more subtle link between the Institute and the Royal Aeronautical Society and our two countries.

I am grateful to your Council for the privilege of being included in their list of scientists and engineers from the United States and from England who, on this date, annually recall the first controlled and sustained flight and present some aspect of aeronautics of topical interest.

Introduction

The majority of lecturers in this series have been scientists, responsible for the direction and control of research, who have indicated the path leading towards improved performance of aeroplanes or power plants. They have presented their work in the knowledge that they serve to educate other scientists working in the same field, directing bodies for the function of their policy and engineer in their application to design.

As a design engineer, the experience I have to report embraces a smaller field. During the last few years The Bristol Aeroplane Company, Ltd., has devoted a considerable proportion of its effort towards the design and development of large transport landplanes with turboprop engines. With this as my subject I am aware that various papers have been published recently dealing with various aspects of the same type of aeroplane and power plant. Their scope covers the field of broad conjecture, deductions along paths of established aerodynamic and thermodynamic principles, and reports on the performance of experimental and development aeroplanes and power plants. For a working basis one appears to be on fairly safe ground if one accepts the conclusion that, for transport aeroplanes, the crossover point in the choice between turbojet and turboprop engines is at medium ranges, with the turboprop giving greater economy at long range.

Probably of all the world's records the one for long distance bears least relationship to feasible transport operation. To achieve it, all normal standards have to be ignored; the take-off is long and hazardous, the aeroplane is overloaded to a point where the factors of safety for strength and performance standards rely on the cooperation of fate, and not less trust is placed in the survival of the motors. The sole restraint is on the patience in awaiting a favourable weather report.

In civil operation one does not find the airworthiness authorities more sympathetic when economic difficulties arise and the pay load inevitably depends on the difference between the gross weight of a fully certified aeroplane and the sum of the unloaded weight and the fuel weight.

The selection of turboprop engines offers one solution to the problem of long range, and the size of aeroplane in which they are installed must be adjusted to suit the load to be carried. With established technical data this is just a case for simple arithmetic.

The most obvious change indicated by general investigations into the effect of using turboprops is a rise in the economic cruising speed and altitude. Some have derived satisfaction from the fact that this increase is consistent with earlier steps, for it can be shown that in the decade 1930–1940 the standard of performance of transport aeroplanes gave cruising speeds of the

* Presented before the Institute of Aeronautical Sciences in the U.S. Chamber of Commerce Auditorium, Washington, D.C., December 17, 1949.

† Chief Designer, Aircraft Division.

order of 200 m.p.h. at 10,000 ft.; in the decade 1940–1950, 300 m.p.h. at 20,000 ft.; and now in the next stage, 400 m.p.h. at 30,000 ft. This seems to clinch the issue.

The review I wish to make is from the engineer's angle, for at these higher cruising speeds and altitudes, together with expanding requirements for range and the increasing size of aeroplanes, new technical problems have been introduced and old ones have been amplified. The role of the designer of transport aeroplanes is to find sound solutions to these problems while still satisfying the needs of the operator by providing him with efficient load-carrying capacity and without departing from those standards of safety established and defined by past experience.

The main section of the paper deals with certain technical aspects relevant to the design of large aeroplanes. To give perspective to this part, two subsidiary sections are included.

The first is a brief review of the characteristics of the turboprop engine immediately available for the type of aeroplane under consideration.

The second is an analysis of the primary aerodynamic characteristics determining operating efficiency and, hence, defining the geometry of the aeroplane.

(I) Turboprop Engine Performance

The turboprop engine possesses, in common with the turbojet engine, the fundamental advantages of rotating as opposed to reciprocating machinery—e.g., freedom from low-frequency vibration and noise, mechanical simplicity making for ease of maintenance and reliability, and the absence of any immediate limitation on the output of a single unit. The first two of these features have so far been amply demonstrated in service.

In these respects the turbojet engine appears to be extremely attractive, offering as it does many advantages from the point of view of the air frame—e.g., grouped power units and lower installation drag. However, the saving thus attained is offset by the relatively poor propulsive efficiency of jet units at the speeds that are appropriate when long range and high pay load are the prerequisites. Also a handicap of the turbojet engine for civil aeroplanes is its comparatively low take-off thrust. If cruising conditions alone are made the criteria, a good case can be made for the turbojet over a considerable range of speed and distance, since the saving in weight is sufficient to compensate for the greater fuel load. However, when take-off performance is a major criterion, assuming no recourse to assisted take-off by special means, then a larger unit is required, and the weight-saving associated with jet propulsion is not so evident. The turboprop, on the other hand, can inherently provide ample take-off thrust, and the power-plant weight may, in fact, be of the same order as that for a turbojet for equal take-off performance.

The comparison between the turboprop and the piston engine is more subtle, since each utilises the same final propulsive means, and the turbine, when fitted with a heat exchanger or using a high compression ratio, has a thermal efficiency of the same order as that of the piston engine.

The differences here mainly concern the operating characteristics of the two engines and are essentially linked with the characteristics of the aeroplane under consideration. For example:

(a) The compression ratio of the turbine engine is not constant but increases with reduced temperature. This leads to an improvement in efficiency with increase in altitude.

(b) The compression ratio and peak temperature are functions of r.p.m.; this results in a deterioration in efficiency at part load.

(c) The ram compression ratio, besides increasing the air mass flow, also enables the working fluid to expand over a higher ratio with consequent increase of power and improved economy.

The three effects combine to make the economic operation of propeller gas turbine essentially restricted to high altitudes, speeds, and powers. As the necessity for operating the engine at a high percentage of maximum power increases the economical latitude and speed of the aircraft, these conditions can be fulfilled simultaneously without detriment to the aircraft performance.

Factors Affecting the Performance of Turboprop Engines

Having established the basis of the turboprop engine, it remains to discuss the methods available for improving its overall performance as an aeroplane power plant. As usual in power plant design, once the fundamental principle has been established and executed efficiently further economy and performance can only be bought at the expense of increased complexity and weight. Thus, a balance between these must be arranged to result in the minimum power-plant weight and specific fuel consumption, with due consideration of other factors that may affect the overall aircraft performance. A fundamental requirement for high efficiency is to reduce the rate of energy rejection in the exhaust, and it is evident that two alternative methods are available. Exhaust heat energy may be recuperated by means of a heat exchanger and fed back into the cycle, or the exhaust temperature may be reduced by increasing the expansion ratio. Another method is to raise the peak temperature, but this is not such a fundamental variable since it is not applicable to the ideal simple cycle. These devices to improve efficiency will be further considered, together with their effect on engine weight.

Heat Exchangers and Other Modifications to the Simple Cycle

The use of a heat exchanger of moderate thermal ratio (about 40 per cent) raises the efficiency of the practical cycle to the same order as that of the piston engine without recourse to a compression ratio greater than 5:1. However, it immediately necessitates ducting the compressed air before combustion into juxtaposition with the exhaust gases, with inevitable increase in bulk and weight of the installation and the creation of parasitic energy losses. A further objection is that exchanger matrices are extremely susceptible to high gas temperatures, which at present prevents the heat exchanger engine from taking advantage of increase in the latter. Despite this, however, there may well be a case for this type of engine in very long-range aircraft, especially when a satisfactory regenerative matrix has been developed capable of thermal ratios of up to 90 per cent.

Intercooling and reheat are well-known devices for improving the efficiency of a gas-turbine cycle, but for aeroplane applications their complexity is likely far to outweigh the small decrease in fuel consumption.

Development of the Simple Cycle

It would seem that the immediate programme of development for turboprop engines should concentrate on improvement of component efficiencies in order to exploit fully the potentialities of the simple cycle when optimum values are prescribed for the main parameters. This approach will result in a light, simple power unit of low frontal area and attractive fuel economy.

It is convenient at this stage to separate the thermal cycle from the means of propulsion, and consider the compressor turbine combination as a gas generator producing "gas horsepower." The best means of dividing the available power between propeller and jet may be examined subsequently.

It should be noted that this procedure is satisfactory for turboprops, since the propulsive efficiency is nearly independent of the thermal efficiency; but this is not so for turbojet engines, and such a procedure may lead to erroneous conclusions.

The specific output and thermal efficiency of the gas generator are functions of four main variables. These are compressor temperature rise (or compression ratio), peak temperature, compressor efficiency, and turbine efficiency; other parameters have fairly well-defined values. Fig. 1 shows the effect of the first two variables for representative values of component efficiency, where thermal efficiency is plotted against gas horsepower for a gas-generator unit. It is evident that it will pay to use as high a gas temperature as possible for maximum power output, and this is also advantageous from the point of view of efficiency, especially at high compression ratios. The peak temperature assumes this importance in the practical cycle because of the fact that the unit would require a power input if no fuel were supplied and the peak temperature were equal to the compressor delivery temperature. A certain fuel input is necessary to overcome the sundry losses before the efficiency attains a positive value; this quantity becomes of less importance as the peak temperature is raised. At extremely high values, we approach the ideal cycle condition where the efficiency is dependent solely on compression ratio. To exploit the advantages of high temperature, we are dependent on advances in materials, or, alternatively, a system of blade cooling must be developed. The latter expedient immediately introduces a compromise, since substantial losses are associated with the provision of a cooling flow. The increased power output resulting in low engine weight is also compromised to a certain extent, and it would seem that low specific weight is more likely to be immediately obtained by aerodynamic and structural refinement of the components.

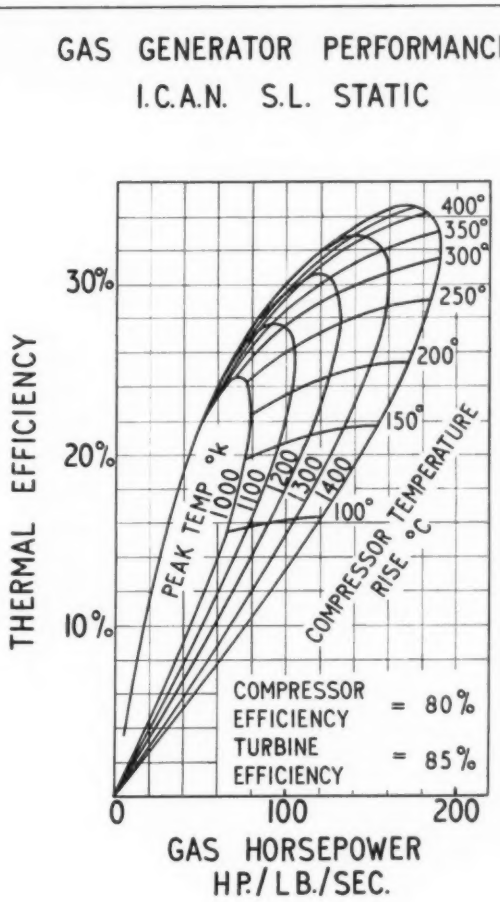


FIG. 1.

For the purpose of arriving at the best type of turboprop for the immediate future, Fig. 2 has been prepared for a fixed value of maximum temperature—i.e., 827°C., which represents the best that can be achieved with current materials without recourse to turbine blade

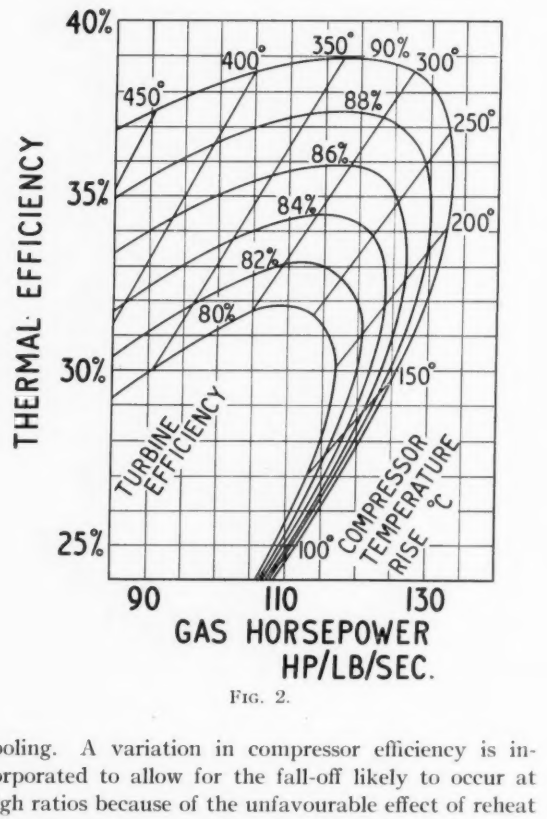


FIG. 2.

cooling. A variation in compressor efficiency is incorporated to allow for the fall-off likely to occur at high ratios because of the unfavourable effect of reheat in the stages and the difficulty of matching them satisfactorily over the operating range. The turbine efficiency is retained as a variable together with compressor temperature rise; the former is not necessarily dependent on expansion ratio, since the beneficial effect of reheat in a turbine tends to compensate for the difficulty in matching. It will be seen that turbine efficiency has a marked effect both on the absolute values of efficiency and output and on the optimum compressor temperature rise.

Turning to the optimum value of the latter parameter, a compromise between take-off and altitude cruise conditions may influence the value chosen. However, for a turbine efficiency of 86 per cent, which is reasonably attainable, it is seen that the optimum temperature rise for efficiency is in the region of 300°C. (9:1 compression ratio), while the value for best power output per unit air flow is just over 200°C. (about $5^1/_2$:1 compression ratio).

Clearly, the question of weight now arises, and the high compression engine will be penalised on this score because of both the lower specific output and the weight of extra compressor stages and the change from light alloys to steel in certain parts of the engine structure to withstand the high temperatures.

Before attempting to evaluate this compromise, it is necessary to convert the power and efficiency figures of Fig. 2 into thrust horsepower and specific fuel consumption. The optimum division of work between propeller and jet may be determined for any one operating condition, but for a specific aeroplane application this also is subject to a compromise with take-off requirements and is governed by extraneous limitations on propeller diameter, speed, etc. In this respect the high cruising r.p.m. of turboprops may be an embarrassment, but the use of an independent power turbine enables the problem of take-off thrust to be overcome by overspeeding while retaining optimum r.p.m. for cruise.

Assuming the use of this device, the equivalent powers and fuel consumption have been estimated for the cruise condition of Fig. 2 and a turbine efficiency of 86 per cent. Together with a curve of estimated weight against compressor temperature rise, this enables Fig. 3 to be drawn.

This is a plot of engine plus fuel weight versus compression ratio for a range of durations, all at 30,000-ft. 350-m.p.h. cruise conditions. It can be seen that the

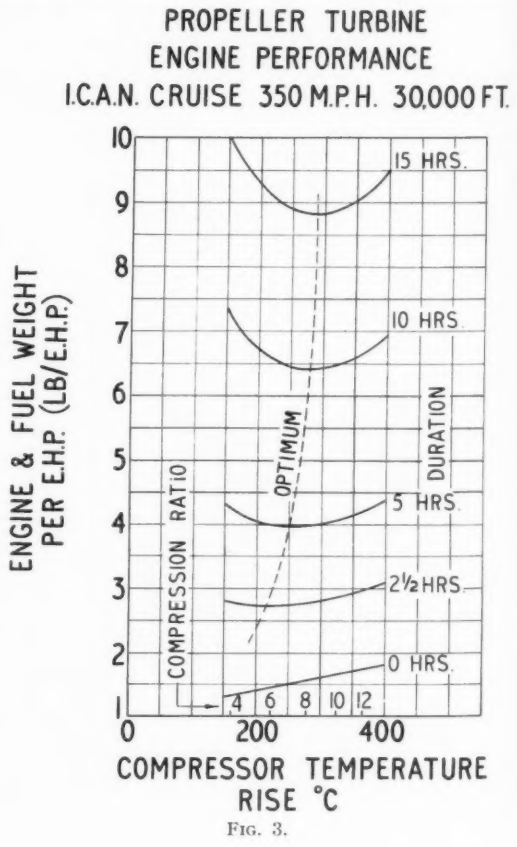


FIG. 3.

minimum engine plus fuel weight is achieved at a compression ratio of about 8:1 for durations above 10 hours. Up to durations of 5 hours, the importance of engine weight predominates, and compression ratio is not critical; low compression ratio engines are then optimum from the point of view of simplicity and initial cost.

The foregoing is still not the complete story on the selection of the turbine-engine parameters. The variation of power and consumption with altitude and speed is affected by the choice of compression ratio and peak temperature. For instance, a high-compression engine is more susceptible to ambient temperature than one of low-compression ratio, so the high-compression engine will show more improvement with altitude and more power loss under tropical conditions. High temperatures, on the other hand, reduce the effect of ambient temperature slightly.

There is as yet not sufficient evidence on the subject of scale effect to decide on the optimum size of unit. Fundamentally, the diameter and length of the blade-carrying elements will vary as the square root of the power for a given design, so the specific weight would be expected to increase with size. However, manufacturing limitations tend to make small engines heavier, and over the present range of propeller turbines the specific weight of similar designs appears to be roughly constant. It does seem probable, however, that component efficiency may be expected to increase slightly with engine size. In certain cases it is beneficial to couple two small engines together—e.g., for the Brabazon Mk. 2 the coupled layout is more compact and more suitable for buried installation than one large engine. In this particular case also, the choice of engine compression ratio is clearly near the thermodynamic optimum, since the range required lays all the emphasis on specific fuel consumption.

(II) Aeroplane Performance

When reciprocating engines are employed and a normal relation is maintained between maximum continuous boost and maximum economic boost, the power available at low altitude (5,000 ft.) at the former condition is severely limited; this arises from the fact that it is desirable to arrange that the full throttle height at cruising r.p.m. is at least as great as the desired cruising altitude, usually in the region of 20,000 ft.

The take-off power is also limited by the basic characteristics of this type of engine, though to a lesser extent than the maximum continuous power.

These conditions will often result in take-off performance and airworthiness performance standards having an influence on the choice of wind loading and aspect ratio; the necessary compromise resulting from consideration of those items of the performance may have an adverse effect on the pay load/range performance.

For aeroplanes with reciprocating engines, the pay load/range characteristics improve with increasing wing loading, and such increases have been made, through increase of weight, during the development of all successful aeroplanes, since means have been found to circumvent critical conditions.

A different situation obtains when turboprop engines are employed; this arises from the fact that the power output of a turbine engine increases continuously as the altitude is reduced from cruising altitude to sea level, and it is unlikely that aeroplane performance at the lower levels will be important to the extent of determining gross weight or wing design. For turboprop-engined aeroplanes, attention will therefore be directed towards attaining the best possible efficiency under cruising conditions.

When turboprops are used, high cruising altitudes are essential; there is no other way to obtain an acceptable pay load/range performance; any other advantages that may accrue are incidental, and any penalties involved are small in relation to the gain in economy and must be accepted. Also, it is essential to operate the engine at the maximum permitted continuous r.p.m. and power in order to achieve the minimum specific consumption, and it follows that at the lower altitudes, where the minimum power required is less than the power available, the equilibrium forward speed is far above the economic speed. The only way to reduce the power available so that an economic condition is obtained is to operate at the highest altitude consistent with the maintenance of sufficient reserve climb performance. This altitude increases as weight is reduced by the consumption of fuel and is rather nearer to the ceiling than to the altitude for maximum speed.

Thus it is necessary to cruise at a speed closer to the most economic speed than when reciprocating engines are used. It follows immediately that the induced drag—and, hence, the wing span—plays an essentially more dominant role when turboprop engines are used.

Cruise Procedure

When the cruising r.p.m. of the engines are maintained constant, the performance of an aeroplane at different weights and over a wide range of altitudes may be presented in the form shown in Fig. 4, the merit of which is that the effect of alternative flight plans on speed, specific range, cruising altitude, and reserve club performance is immediately apparent. This form of presentation was devised by the B.O.A.C. and was first given by Dykes in his paper at the Second I.A.S.-R.Ae.S. Conference.[5]

Table 1 compares the result of following various flight plans, since the weight of the aeroplane falls because of the consumption of fuel. The initial weight is not important in this comparison and has been taken

TABLE 1

Flight Plan	Mean Cruising Speed, M.p.h.	Still-Air Range, Miles	Fuel for 5,000 Miles	Difference from Plan (3)	Maximum Altitude, Ft.
(1)	358	4,900	$0.386W$	$+0.048W$	33,000
(2)	319	5,900	$0.328W$	$-0.010W$	39,000
(3)	342	5,700	$0.338W$	.....	39,000
(4)	348	5,100	$0.372W$	$+0.034W$	30,000

as 300,000 lbs. The still-air range (column 3) assumes that the fuel available amounts to 37.5 per cent of the initial weight. The flight plans examined are: (1) maximum cruising speed throughout; (2) speed for maximum range throughout (nearly zero potential rate of climb); (3) altitude such that the potential rate of climb is 200 ft. per min.; (4) as in (3), except that the maximum altitude is limited to 30,000 ft.

[An additional plan would be to fly at constant E.A.S. throughout a flight; depending on the speed chosen, the result lies somewhere between Plans (2) and (3).]

It is seen that with one exception the differences between the mean speeds are relatively small; on the other hand, the available range with a given quantity of fuel or the weight of fuel required to fly a given distance varies in a most significant way; for, if the pay load of a fully equipped civil aeroplane be taken as $0.10W$ for Plan (3), then for Plan (1) it would be only one-half that amount.

The reduction in pay load or range due to adopting Plan (3), instead of Plan (2) (which gives the greatest range), is seen to be small, whilst giving a useful increase in speed. Plan (2) would probably not be acceptable because of the small potential rate-of-climb available, whereas a rate-of-climb of 200 ft. per min., available if speed is reduced, should be sufficient to maintain a consistent flight path. The conclusion is that the best plan is No. 3, flight at constant potential rate of climb. It may be noted that this corresponds also to almost constant lift coefficient.

If towards the end of a flight it is necessary to restrict the altitude to, say, 38,000 ft., for reasons of limitation of cabin differential or due to the performance of the air-conditioning system, the penalty is small.

Plan (3) will be used for subsequent investigation of the effects of other variables on long-range operations including the effect of engine failure.

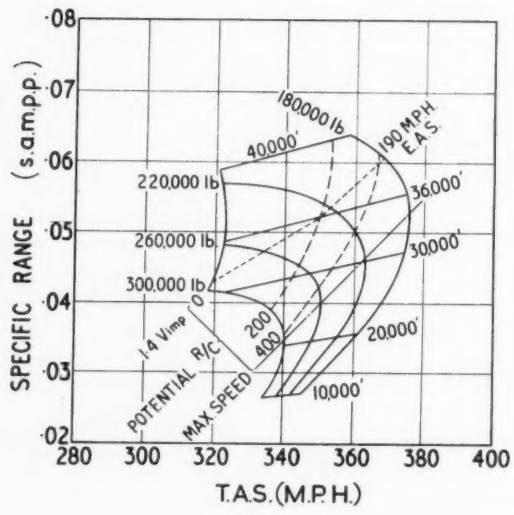


FIG. 4.

Initial Weight

Fig. 4 can also be used to show the effect on range of variation of the initial weight for the particular aeroplane to which it relates. Starting at a series of different weights, the fuel consumed in flying different distances can be determined. A plot of the results would show that, for a wide variation of the initial weight, covering all likely values, the fuel consumed for a given distance is a constant percentage of the initial weight.

This result is also indicated by the Breguet range formula, which has as its basis flight at constant life coefficient.

It is clear from Fig. 4 that the mean cruising speed is sensibly independent of initial weight.

Thus, for a fixed geometry of aeroplane, engine power, and set range, the potential pay load is approximately a constant percentage of the gross weight, such tendencies as there are being favourable with an increase of gross weight in spite of the lower operating altitudes.

Effect of Temperature

If an aeroplane is required for world-wide service, it is necessary to take account of the effect of temperature on both the power output of the engines and on the power requirements of the aeroplane.

The influence of ambient temperature on the performance of a typical aeroplane with all engines operating is shown in Fig. 5. The three full lines show specific range in ICAN Standard conditions and in conditions approximating to temperate maximum conditions (15° and 26°C., respectively, higher than ICAN Standard). The dashed lines refer to cruising altitudes and cruising speeds are marked on the full lines.

In this example the cruising altitude comes down by about 3,000 ft., and the range and speed by approximately 5 per cent, for each 10°C. increase above ICAN Standard. For temperatures lower than ICAN Standard, the performance is, of course, improved to a corresponding extent.

Thus it appears that the ambient conditions in flight have a greater influence on the performance of aeroplanes with high-compression turboprop engines than has been the case with reciprocating-engined aeroplanes.

A partial solution to the problem this presents is to permit an increase in the cruising r.p.m. of the turbines under high ambient temperature conditions; the engine manufacturer and those responsible for certifying approval of his product are then set a difficult task in formulating a representative type test. The stress conditions in a turbine engine are complex and depend on blade temperature, gas loading, and r.p.m.

At the other end of the scale, it may be necessary to restrict the r.p.m. in substandard temperatures.

Investigations show that the relation between potential rate-of-climb and cruising lift coefficient is little affected by change of atmosphere, and, as constant potential rate-of-climb implies almost constant lift coefficient, the selected flight plan can be followed by the crew in any atmosphere from the indications of an incidence meter; too large an incidence would imply too great an altitude and vice versa.

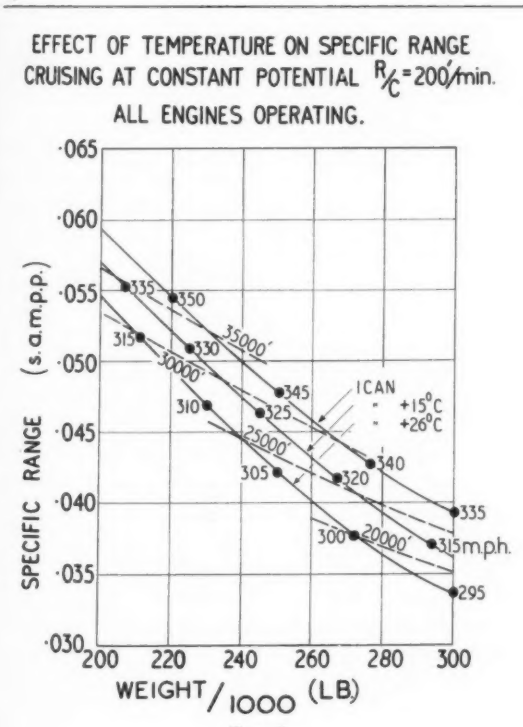


FIG. 5.

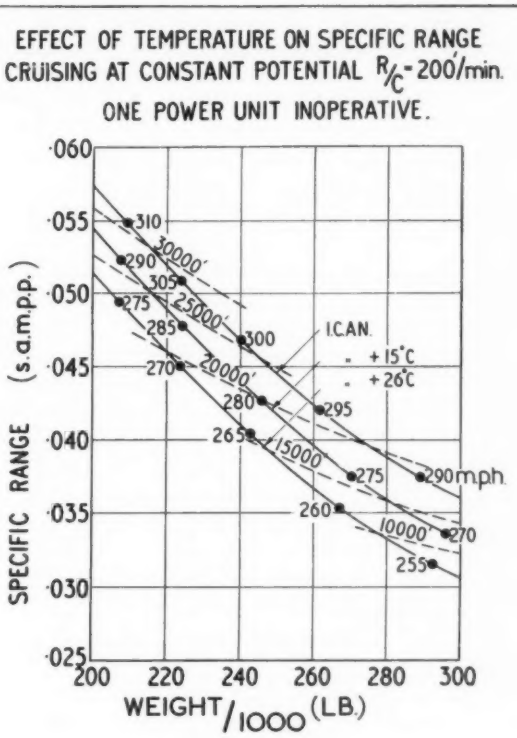


FIG. 6.

Engine Failure

Certain performance requirements in a high ambient temperature after the failure of an engine may be an important design criterion. This can arise if the air-line operator demands that a certain minimum altitude must be maintainable, a necessary precaution during certain climatic periods or conditions, such as the monsoon, when the phrase "flying above the weather" has more than usual popular appeal. Such requirements are of some importance in the initial selection of the engine characteristics, for a turboprop engine designed as an optimum for ICAN Standard conditions will not necessarily be the best design if operation through the tropics is involved.

Fig. 6 shows the performance of the aeroplane to which Fig. 5 relates after the complete failure of a power unit, and Table 2 brings together typical values from both figures to show the effect of power-unit failure; the effect of ambient temperature is also

TABLE 2

	Atmosphere	ICAN Standard	ICAN +15°C.	ICAN +26°C.
All Engines Operating	Mean speed (m.p.h.)	350	332	313
	Mean altitude (ft.)	36,000	32,000	28,000
	Mean specific range (relative)	100%	95.5%	90.5%
One-quarter Engines Inoperative	Mean speed (m.p.h.)	307	287	271
	Mean altitude (ft.)	30,000	25,000	20,000
	Mean specific range (relative)	95.5%	90.5%	85.5%

brought out clearly. The values given relate to the performance during the period of consumption of the last half of the fuel carried. It will be seen that the effect of power unit failure on range is the same as the effect of a rise of 15°C. in ambient temperature but that the power unit failure results in a much greater loss of speed.

Wing Design

Having found that the turboprop aeroplane must be operated closely to a prescribed cruising procedure, the effect on operating efficiency of varying the wing area and the aspect ratio can be examined. The actual selection of these values will, of course, depend on a balance between aerodynamic gains and structural penalties.

Thus a limit is imposed on span by the amount of structural stiffness than can profitably be provided to inhibit catastrophic aeroelastic effects; this is a function of the proportions of the wing box, which are in turn determined by the aspect ratio. At the relevant speeds there are major technical obstacles to designing a wing of very high aspect ratio without incurring an uneconomic weight penalty.

As primary technical considerations do not impose a limit to wing loading, the effect of this variable on pay load range efficiency can be examined at constant aspect ratio without invalidating the conclusions.

The effect of a change of wing area on a given aeroplane is to displace the selected cruise procedure line both horizontally and vertically, the cruising speed rising and the specific range falling with increase of wing loading. Table 3 gives a summary of the results of varying the wing area at constant aspect ratio.

TABLE 3

Initial Wing Loading	Mean Speed, M.p.h.	Mean Miles per Lb.	Fuel for 5,000 Miles	Range of Cruising Altitude, Ft.
45	333	0.0500	$0.333W$	29,000–42,000
55	341	0.0480	$0.348W$	26,000–41,000
65	349	0.0463	$0.362W$	24,000–40,000
75	355	0.0445	$0.376W$	22,000–39,000

It is seen that a reduction of wing loading raises the cruising altitude and, hence, the economy, but the amount of the improvement is comparatively disappointing. The reason is that the increase in drag coefficient has a large detrimental effect. As will be shown later when allowance is made for the different wing structure weights at the several loadings, it is apparent that there is no optimum value of the initial wing loading in relation to pay load range performance. Over the range of loadings tabulated, the sum of the weights of the fuel required and of the wing structure is substantially constant.

Purely from performance considerations, an increase of aspect ratio can be expected to have a more favourable effect, for a reduction in induced drag leads to both higher altitudes and higher cruising speeds. Table 4, in which the aspect ratio is varied at constant wing area, summarises the results.

TABLE 4

Aspect Ratio	Mean Speed, M.p.h.	Mean Miles per Lb.	Fuel for 5,000 Miles	Range of Cruising Altitude, Ft.
8	340	0.043	$0.388W$	23,000–37,000
10	345	0.048	$0.348W$	26,000–39,000
12	350	0.054	$0.309W$	30,000–41,000

Power Loading

While turboprop engines are in their present stage of development, the power produced and the specific fuel consumption achieved at the time of initial service may differ from the values used in the aeroplane design calculations.

A change in turboprop engine power may be considered in two stages, the effect at a constant gross weight and the effect of the gross fixed weight can be increased.

At a fixed initial weight, the primary effect on range and pay load of an increase or decrease in engine power is, perhaps surprisingly, nil. The truth of this is best demonstrated by the following simple analysis:

At constant C_L, C_{D_T} is constant and σV^2 is proportional to W, or

$$V \propto (W/\sigma)^{1/2}$$

Hence $D_T W$ and d.hp. (or e.b.hp. if propeller efficiency is constant)

$$\propto (W^{3/2}/\sigma^{1/2})$$

Hence, assuming that the specific fuel consumption is constant, lbs. per hour. $\propto (W^{3/2}/\sigma^{1/2})$.

Then specific range (= V/lbs. per hour) $\propto (W/\sigma)^{1/2} (\sigma^{1/2}/W^{3/2})$ or $\propto W^{-1}$, so that, for constant specific fuel consumption and the selected flight plan, the specific range is a function of weight only. In fact the specific fuel consumption falls slightly with increasing altitude, and, if an increase in engine power at a given altitude can be effected at constant specific consumption, the provision of greater power will result in a small improvement to the specific range. If, on the other hand, the provision of greater power is accompanied by a small general reduction of specific consumption, the value of the extra power in respect of range will be more marked. Where long range is of paramount interest, a moderate reduction in the specific consumption is of more value than a large increase in power.

It is not so easy to generalise on the value of an increase in power in improving range or pay load via an increase in all-up weight, since the magnitude of the increase, the characteristics of the particular aeroplane concerned, the question of the provision of sufficient structural strength at the increased weight and with the new distribution of weight, and a whole series of

Airworthiness and Operational requirements are involved.

It is clear that the circumstances are quite different from those obtaining when piston engines are employed. Additional power to permit an increase of gross weight by overcoming some critical operating condition and thereafter throttling back to a more favourable power for fuel consumption in cruising flight is no longer possible. Also, in the example quoted where one engine was inoperative, fuel is not saved by deliberately shutting down an engine or group of engines.

It appears that for any long-range aeroplane the pay load will be substantially a constant percentage of the gross weight and that the only features of prime importance in a gas-turbine engine are the installed weight and the specific fuel consumption.

(III) Analysis of Structural Characteristics

Dynamic Effects

On being faced with the design of an aeroplane of gross weight far beyond our previous experience, some thought was given to the need for including certain fundamental effects that had hitherto not been considered in normal methods of design. With greater size, greater masses, and greater structural deflections, it was apparent that the dynamic response under the action of rapidly applied external forces could produce stresses different from those estimated by the usual static analysis. For this investigation a knowledge of the behaviour of structure in vibration was required.

Normal Modes of Vibration

Calculations involving dynamic response are simplified by the use of the normal modes of vibration, since elastic and inertia couplings are absent and, for the purpose required, aerodynamic coupling is usually so small that it can be ignored. It should be remembered however that the use of these modes is arbitrary in the sense that the elastic and inertia terms are combined irrespective of the aerodynamic terms. As the speed is increased the inertia-elastic coupling becomes progressively of less importance than the aerodynamic-elastic coupling, so that a point may be reached at which it is preferable to use a series of aerodynamic-elastic (i.e., divergence) modes. However, for conventional subsonic aircraft the normal mode approach is likely to be the more accurate.

The most important property of a normal mode is that, so long as structural damping is absent (a valid assumption for aircraft), its shape is fixed for any given inertia and elastic system. So-called resonance modes, obtained by applying an oscillating force, are composed of a number of normal modes, one of which predominates. The contributions of the lesser modes must be removed before a true normal mode can be determined. The three methods of determining the normal modes of vibration of an aircraft structure—namely, by calculation, model tests, and full-scale tests—will be discussed.

Calculation of Normal Modes

The inertia and elastic forces that determine the shape and frequency of a normal mode can be related by a simplified form of Lagrange's equation[6] for each coordinate.

Since an aircraft structure is continuous, the number of generalised coordinates and, hence, the number of equations are infinite. An approximation to the elastic and inertia properties of a structure must be made by considering a limited number of discrete masses connected by flexible but massless rods. The number of degrees of freedom (or the number of masses) chosen depends on the complexity of the structure and the number of modes to be determined. If only the first or fundamental mode of a wing is required, it is normally sufficient to consider six degrees of freedom (i.e., three masses per wing), but if four to six normal modes are needed, about 20 degrees of freedom must be considered. Since the work involved in the calculation is roughly proportional to the square of the number of degrees of freedom, the computing time for six modes is about ten times that required to determine the fundamental mode only.

Having idealised the aircraft structure into the required number of masses and stiffnesses, the equations of motion are solved by an iterative process. A mode of vibration is assumed, and the inertia forces corresponding to this mode of vibration are determined. These inertia forces when applied to the elastic system will give a deflected mode in general different from that assumed originally. The process is repeated until the assumed and derived modes coincide. At each stage the modes are made "free-free"—i.e., the centre of gravity of the aeroplane is given a displacement such that the total energy of the system is zero. For modes above the first, the contributions of all modes lower than the one considered are removed during iteration by applying the orthogonality condition.

This method has been in use for some years at The Bristol Aeroplane Company, Ltd. The computing time has been shortened by the use of influence coefficients for the displacements due to inertia forces; more recently, it has been further shortened by the use of matrices and, although the matrices involved may be of the 20th order, the method leads to quicker and more accurate computing. The development of electronic calculating machines may eventually reduce this computing time even further and enable more detailed calculations to be undertaken.

Whereas in calculating the first few modes of vibration it is sufficient to consider the flexural and torsional stiffnesses of the wing and the flexural stiffness of the body in connection with the masses, if modes above the third are required, the effects of shear deflection and

shear lag must be taken into account. In a particular application, the inclusion of shear deflection in the calculation reduced the frequency of the second uncoupled flexural mode by about 5 per cent, the third by 13 per cent, the fourth by 21 per cent, and the fifth by 30 per cent. The corresponding reduction in the frequency of the coupled modes was about 15 per cent on the fifth mode. The shape of the modes was also affected but not to such an extent as the frequencies.

In the case of fuselages where shear stiffness is a greater proportion of the total stiffness, the reduction of frequency is even greater. Again, in a particular application[7] the frequency of the first mode was reduced by 8 per cent, the second uncoupled flexural mode by 34 per cent, the third by 49 per cent, and the fourth by 60 per cent. Fortunately, the effect of shear deflection can be included in the calculations without much difficulty by modifying the matrix of influence coefficients.

The result of including the effect of shear lag across the wing box[8] and around the body periphery is again a reduction of frequency, the reduction being of the same order as that due to shear deflection. However, shear lag seems to have little effect on the shape of the modes, and it is probably sufficient to allow for its effect by a final adjustment to the frequencies.

The introduction of shear deflection and shear lag, together with a greater number of degrees of freedom, including perhaps engine nacelle and tail-plane flexibilities, makes the calculation of the higher modes of vibration a lengthy process. The sixth mode is probably the upper limit to which calculation can normally be applied. For the higher modes there is a distinct advantage in the use of a vibration model.

The Vibration Model

By building a model that reproduces the elastic and inertia properties of the aircraft to some suitable scale and by applying an exciting force to this model, it is possible to measure the frequencies and the shapes of the normal modes in much the same way as the modes are derived from resonance tests on complete aircraft.

Most vibration models used so far have been of a simple design. The flexural and torsional stiffnesses have been simulated by suitably shaped bars to which concentrated masses are attached at a few stations. The number of discrete masses can be somewhat higher than the number used in calculations, with consequent improvement in accuracy. However, the reliability of the model results do depend on the assumption that the elastic properties of the aeroplane can be estimated accurately. This is a matter of some difficulty, especially if the structure includes cutouts or other discontinuities. Also, in the case of sweptback wings where the old methods may no longer suffice, the time required to apply the more comprehensive theories is considerable. However, with increasing performance and the consequent tendency towards overriding aeroelastic criteria, it may well be necessary to devote this time to obtaining reliable values of the elastic properties of the structure at an early stage.

A vibration model must be supported in such a way that the "free-free" modes are obtained, and this is usually done by suspending the model on long rubber cords such that the rigid body frequency of the model is small compared with its lowest fundamental frequency. A ratio of less than one-third is necessary if there is to be no interference of the modes regardless of the position of the points of support.

As with the calculations so with the model, the accuracy of the results clearly depends on reproducing the relevant stiffness characteristics and, again, if only the first three modes are required, the model can be relatively crude. If accurate knowledge of higher modes is required, there seems to be no alternative to building a fairly complete scale model of shell form such that shear deflections and shear lags are reproduced. When a high degree of geometric simularity is maintained, there is, of course, no need to indulge in complicated estimates of structural stiffness.

This type of model would be more difficult and costly to make owing to the limitations of the minimum possible thicknesses of important members. Increasing the length, mass, or frequency ratios or reducing the modulus of model to full scale does not change the order of the overall problem, which becomes more difficult the larger the aircraft.

Thus, while such a detailed model may be justified for certain research investigations, it is not suited to routine design analysis. For this purpose it is necessary to rely on experience to modify the results obtained from a simple model or to carry out calculations that include the factors appropriate to the modes required.

Resonance Tests on Complete Aircraft

It has been the practice for some time to carry out ground resonance tests on prototype aircraft prior to their first flight.[9] This serves as an experimental check on their flutter and vibration characteristics. This is a late stage to discover, for the first time, that some intrusive mode asserts itself in embarrassing company.

Even the results of ground tests cannot be accepted without reserve, for the stresses induced by the vibrations are often insufficient to promote the buckled state of the skin that may be appropriate to flight conditions. However, I am not aware of an instance where this discrepancy has been a just reason for optimism in critical cases.

As with the model, the aircraft must be supported in such a way that there is no interference with the lowest frequency modes.[10] This condition becomes increasingly difficult to meet as the size of the aircraft increases. The simple expedient of reducing the pres-

sure in the tyres of the landing wheels is satisfactory only for small aeroplanes.

Some aircraft have been supported on very low-pressure air bags with varying degrees of success. Good results have been obtained in the higher modes, but with large aeroplanes the lower modes have been quite unreliable. The bags so far developed have a defect in that they have either inadequate lateral shear stiffness or too great a vertical stiffness. A more promising method is to use high-pressure low-friction liquid springs with a large reservoir. These are capable of giving very low frequencies and can be used to support the aircraft at the jacking points.

The most simple way of providing excitation is by an inertia vibrator producing sinusoidal forces, although, unless the speed of rotation is maintained constant, harmonies may be present at very low frequencies. Moving coil exciters, in which a coil attached to the aeroplane is supplied with variable frequency alternating current and vibrates in a strong magnetic field, have the important advantages that by employing a number of exciters, coupled electrically to give phasing, the separation of particular modes is facilitated. Excitation at low frequencies is still a problem with the electric exciter, which has the further disadvantage that an external rigid mounting is necessary. Lateral excitation is thus more difficult. An accurate steady speed control is essential for either type of exciter, since there is a tendency for the speed to seek and remain at the frequency of a powerful mode.

Amplitudes of vibration are usually measured by electrically integrating velocity or acceleration. The velocities pickup is more accurate at low frequencies, since the acceleration pickup response, being proportional to frequency, tends to exaggerate any harmonics that may be presented in the motion and suppress the fundamental. The acceleration pickup, however, has the same advantage as the inertia type exciter—i.e., that it requires no external rigid support and it may be easily fixed to the aircraft in any position. Coupled with an efficient band-pass filter to remove harmonics, the acceleration pickup shows distinct advantages over other types. The accurate measurement of phases is the most necessary (and the most difficult) part, since the motion will be a combination of several modes.[6] One method is to use a double-beam oscillograph for comparing the wave traces from the pickups and exciter. Another is to use a single-beam oscillograph feeding in the two-wave traces in perpendicular directions to produce a Lissajous figure from which the phase can be deduced. Both methods become inaccurate if strong harmonics are present in the wave traces, as when there is motion in a plane perpendicular to the axis of the pickup. The most accurate is to record continuously the vibration on a film and measure the phases. The use of a multichannel amplifier and recorder enables an accurate recording of the amplitude, frequency, and phase of the vibration of a dozen or more pickups to be made simultaneously. Analysis of this film on the drawing board is less liable to errors than direct measurements off a flickering oscillograph display. As so frequently happens, the best method takes the longest time.

Flight Resonance Tests

The fundamental symmetric and antisymmetric normal modes can be found by disturbing the aircraft in flight (e.g., by applying sudden elevator or aileron movements) and recording the subsequent oscillation. The presence of aerodynamic stiffness, damping, and inertia will modify the normal mode, and a correction must be applied to the results to produce the true "vacuum" modes. Higher modes require an inertia exciter and, being more heavily damped, may be difficult to determine. In any case, the use of an exciter is dangerous if there is any possibility of flutter at the exciting frequency. The method will thus usually be restricted to the first two modes where it has distinct advantages (in particular the absence of support difficulties and the more normal loading of the structure) over the ground resonance test.

Dynamic Response

In a structure, if the time of application or variation of the external loads is of the same order as the natural periods of vibration, the effects of flexibility can be important. In the case of a simple mass system an appropriate rate of loading will produce a maximum spring force twice that which would occur if the load were applied slowly. Also, for a harmonic force applied to the same system, the general solution to the equation of motion provides a condition by which, where the force equals a natural frequency of the system, the forces may be limited only by damping in the system and by which, where the time of application of a transient impressed force is of the same order as a natural period of vibration of the system, the forces may be increased by up to 100 per cent. Thus, for calculating the loads in a flexible aeroplane, knowledge is required of the natural frequencies and modes (or the corresponding deflected shapes) in free vibrations. The description of each mode will depend on the relative importance of flexure and twist in its component parts.

If wing and body flexure and wing torsion only are to be considered, the displacement of any point on the aircraft can be expressed in terms of two quantities: (1) the vertical displacements, along a fore-and-aft centre line or along a lateral axis, corresponding to unit displacement in that mode and (2) the rotations of wing sections about the lateral axis for the same unit displacement. The shear, bending moment, and torque in the wings for unit amplitude of vibration of the aircraft in any mode can then be calculated.

In a free vibration, the "inertia forces" at any time are equal and opposite to the "elastic forces." Therefore, given the displacement of the reference section in

any mode, the elastic effects may be found as those due to the inertia forces arising in an equivalent vibration of amplitude equal to the displacement. These inertia forces can be used in conjunction with displacement at the reference section to give the equivalent loading on the wing or body at any time during or after a given set of forces.

By considering the displacements defined by the mode considered, the inertia forces can be integrated inwards from the wing tip to produce the inertia shear, bending moment, and torque. These unit values may be tabulated or presented graphically.

The response of any given mode (i.e., one degree of freedom) to an applied external force is found in a similar way to the response of a single mass-spring system. The basic relations are identical, since the displacements at all points in the structure in a given mode are fixed by the deflection at the reference section. The strain energy is equal to the work that would be done by slowly applied forces that produced the actual deflection at the reference section; it is also equal to the maximum kinetic energy of free vibration (of the same amplitude) in that mode.

Thus, to find the "static response" of the nth mode to slow application of a force P, we write

$$\delta n = P\, C_n / E_n$$

where δn is the amplitude of the mode at the reference section, (PC_n) is the generalised value of the force, and E_n is the generalised elastic stiffness of the mode.

If the force P is applied at some place and the structure is constrained to deflect with unit amplitude at the reference section in the nth mode, then the displacement at the point of application and in the direction of the force is defined as the "influence coefficient" C_n of that force in the nth mode.

If the forces are applied slowly, the deflection in each mode gradually builds up to a constant value. When these values are reached, all parts have equal accelerations, and the deflected shape is given by the sum of the deflections in all possible modes. But for normal structures where the masses and stiffnesses are not easily separated, there is an infinite number of modes. While in such a case it is not possible to calculate the deflected shape by using these modes, the effect of rapidly applied or varying forces on the deflected shape and on the internal loading can be found by considering different modes separately. The "static" deflected shape of a structure under uniform acceleration may be simply found as the elastic deflection under a balancing system of applied and inertia forces.

With a spring mass system, if the variation of force with time is known, the increments of momentum and the velocity changes to the mass can be integrated so that displacements due to successive impulses can be found. The free vibrations set up produce displacements in excess of the static deflection and the amount of this excess is called the overswing. The static force that would be required to produce this deflection is called the "equivalent static force," and the difference between this and the impressed force at any time is called the "true dynamic loading."

The loading applied to the structure (i.e., inertia shear, bending moment, and torque) can be found by multiplying the equivalent values as calculated for unit amplitude of vibration by the total deflection referred to the datum. Separating this deflection into its two parts, the static deflection for all modes of vibration is identified with the inertia loading due to the acceleration of the centre of gravity produced by the external forces at the time. The overswing for all modes gives the total dynamic loading. In practice, the dynamic effect becomes small for modes of high frequency.

It may be found that the maximum nett values of shear and pitching moment on the aircraft are not zero, in which case there are errors in the position of the reference datum. These nett values may be used to produce linear and angular accelerations such that the final resultant inertia forces and moments are made zero and the "free-free" condition is satisfied.

Applied Forces

Applied unsteady forces to an aeroplane in flight may arise from either an intentional change in the flight path or from atmospheric disturbances. Taking the case of an aeroplane flying into an uprising current of air, the buildup of aerodynamic loads is complex. During the time the aircraft is accelerating vertically, the vertical velocity thus acquired reduces the effective gust velocity. A certain amount of pitching may also occur, depending on the stability characteristics of the aeroplane, though this is not a major effect perhaps as regards the tail plane.

The Wagner effect is[11] also important. A sudden change of incidence does not immediately produce the lift appropriate to steady-flow conditions; there is a time lag during which the system of vortices is built up. The buildup is asymptotic, and a measure of the rate is such that the wing must traverse a distance of about three wing chords to build up 80 per cent of the full lift increment. By slowing down the application of the gust load, this lag acts as an increase of gust gradient distance by allowing more time for vertical velocity to accrue.

A theoretical solution giving the lag for a sudden change of incidence in two-dimensional flow has been worked out by von Kármán and Sears[12] and for the lift in a sharp-edged gust by Sears and Sparks.[13] Further work by Jones[14,15] deals with wings of finite aspect ratio.

The Gust Case

In a gust the three main factors—viz., dynamic response, vertical motion, and Wagner effect—while

complicating the problem on their own, suffer as well from considerable coupling. A short list of these couplings is as follows:

(1) The change of incidence due to vertical motion of the aeroplane does not produce an alleviating air load immediately.

(2) The flexural response of the wing is aerodynamically damped, and this is subject to lag.

(3) Dynamic response in the torsional modes causes wing twist, which modifies the lift.

(4) The free vortex sheets associated with the transient changes in lift modify the lift on the tail plane.

These and many other interactions must be considered when a rigorous analysis of the behavior of an aeroplane in a gust is undertaken.

In view of the complicated nature of the aeroplane response, it is not surprising that reliable evidence on gust shapes and velocities is lacking. Two approaches to the design methods can be employed.

In the first case, let us suppose that a certain test aeroplane has accumulated sufficient V-g records to establish statistical data. It is then possible to specify that at a given speed a certain value of the normal acceleration can be used as a design value to ensure safety.

Now, if another aeroplane were being designed—provided that the Wagner lag, aeroplane response, and dynamic effects were the same as those of the reference aeroplane—it could be designed to an arbitrary sharp-edged gust and the lag and response effects could be neglected. This approach is not adequate in all cases, since it can be demonstrated that the value of the wing loading has an important effect on vertical response. However, a simple correction can be made by analysing the response with different wing loadings in a sharp-edged gust, and the alleviating factors thus obtained can be correlated to the approved standard. This procedure does not allow for differences in Wagner lag or in dynamic response, which may be important.

In the second case, the analysis is more detailed. Returning to the test aeroplane, if the V-g recorder is so placed that structural accelerations due to dynamic response are negligible, a calculation can be made for a specified gust in which aeroplane response and aerodynamic lags are allowed for. The question is then, what should the gust shape and value be? The evidence published so far is not convincing.

British design practice is to assume a vertical velocity of gust of 50 ft. per sec. E.A.S. at cruising speed. A range of gradient distances is considered, and the maximum velocity is sustained. The assumed gust shape must be used in the response checks of the reference aircraft. If a different shape of gust is taken, a different velocity must be used to produce the same acceleration. Such variations will produce negligible differences to the vertical accelerations to be used in design.

The design cruising speed for a civil aeroplane is currently fixed at a somewhat arbitrary figure, which is taken as constant E.A.S. with altitude, the limiting altitude being that at which this speed is equal to the maximum potential cruising performance of the aeroplane; thereafter it falls with increasing altitude because of engine power limits. Furthermore, the vertical gust velocity (E.A.S.) is reduced above 25,000 ft. in the current ICAO requirements. Consequently, the critical altitude for gust stressing is seldom higher than this figure. For constant E.A.S. cruising and gust speeds, the intensity of gust loading increases with altitude because of the increase in compressibility factors. In the case of an aircraft having V_c equal to 300 m.p.h. E.A.S.) up to 25,000 ft., the increase of gust intensity due to compressibility may be 20 per cent above the sea-level value.

Statistically, an aeroplane will enter a gust asymmetrically more often than not. The head-on collision produces the highest accelerations. If a rolling motion is set up by an oblique intercept, although the downward moving wing aggravates the conditions when it enters the gust, the overall effects are compensated by the effectively greater gradient distance.

Application in Design

These methods of detailed gust analysis, though laborious to apply, give the designer a clearer indication of the characteristics of a particular case and may offer scope for alleviating some of the worst effects arising from gust loads on flexible high-speed aeroplanes.

Such calculations for the gust and landing cases have been carried out by The Bristol Aeroplane Com-

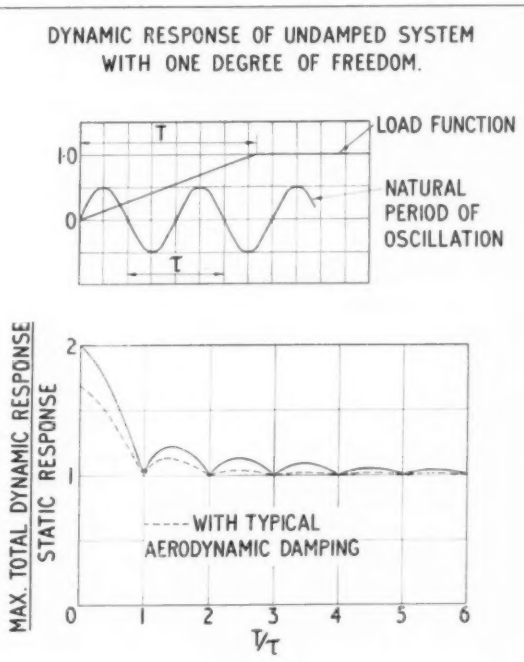


FIG. 7.

BRABAZON I

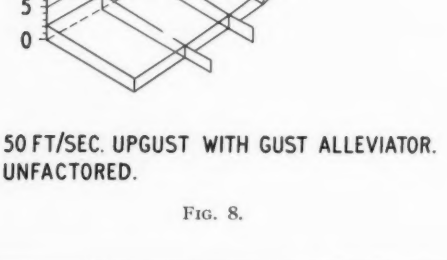
PEAK ACCELERATIONS IN WINGS DETERMINED BY MODAL DYNAMIC RESPONSE CALCULATIONS

50 FT/SEC. UPGUST WITH GUST ALLEVIATOR. UNFACTORED.

FIG. 8.

BRABAZON · I

PEAK ACCELERATIONS IN WINGS DETERMINED BY MODAL DYNAMIC RESPONSE CALCULATIONS.



12 FT/SEC. SYMMETRIC LANDING, $\lambda = 0{\cdot}8$. UNFACTORED.

FIG. 9.

pany, Ltd., on all types since 1944. The following general effects appear from the interactions between dynamic response and damping:

Fig. 7 shows the response of an inertia-elastic system having one degree of freedom to an applied load whose time history resembles the assumed gust shape. The system is first assumed to be undamped. The static response is that which would accrue if the system had no mass; it represents the asymptotic condition where the natural frequency tends to infinity. The maximum dynamic response is the peak value of the actual response, which does not necessarily occur at the same instant as the maximum applied load. It is clear that, for loads whose time of application is less than the natural period of vibration of the system, the amplitude can be up to 100 per cent greater than would be found by neglecting dynamic effects.

In fact, the aeroplane modes excited in a gust will have aerodynamic damping, and this damping is most important for the higher modes. The results so modified are also shown in Fig. 7. The indication is that an important simplifying assumption can be made; the dynamic overswing of modes having T/λ greater than 2 can be neglected.

The procedure then is to treat the problem as if it were static and then add the difference between the dynamic and static responses of the lower modes of vibration of the aircraft.

Figs. 8 and 9 show the maximum transient accelerations in the wing of a particular aircraft due to its dynamic response to gust and landing cases. It is noteworthy that the transient accelerations can be as much as six or seven times the steady values; this large magnification is due to the interactions of the large numbers of degrees of freedom.

At any given altitude the higher cruising speeds associated with turbine engines will increase the severity of the aerodynamic loads due to the gust. There is, however, an important alleviating effect, in that the aerodynamic damping for wing flexure increases as forward speed increases, and this reduces the total response in the fundamental mode. In fact, at high speeds it may make the system "dead-beat" or better.

This again is similar to a simple spring mass system in which the load rises rapidly to a maximum and then falls off in intensity. If sufficient damping of the system be present, then the maximum amplitude and therefore the stress induced in the system can be less than would be induced by the same maximum load statically applied; started otherwise, the maximum stress induced can be less than that calculated assuming the system rigid.

For an aeroplane the damping depends on the dynamic head, the compressibility factor, and the effective change of incidence due to vertical velocity of vibration. It can be found that damping will decrease in importance with increasing altitude and increase rapidly with increasing speed.

These points are illustrated in Fig. 10, which shows the results of calculations carried out on an aeroplane of 230-ft. span and 300,000-lb. gross weight. The results were obtained by step-by-step methods taking into account Wagner effect, freedom of the aeroplane to move vertically and to pitch, aerodynamic damping, and stiffness but neglecting "virtual inertia" effects. The conditions are for a gust of 50 ft. per sec. E.A.S. vertical velocity and a gradient distance of 100 ft.

The points of interest are: (1) The general shape of the curves for the aeroplane are similar to those of a

simple system. (2) Since each of the curves is proportional to the applied loads of its own case, the more rapid fall-away after reaching the maximum illustrates the greater importance of damping at higher speeds. (3) At 370 m.p.h. the flexible aeroplane actually experiences less stress in the gust than would be the case if the gust were considered a static problem.

Hence, the highest speed associated with a given gust intensity will not necessarily give the worst design conditions, and it would appear that, as cruising speeds for large aeroplanes rise, the importance of the "gust at cruising speed" will diminish.

It is interesting to investigate the general trends of the effects of varying size and speed on aeroplane response to gust conditions. Simple physical arguments can be used to demonstrate the effect of the variables.

To give a picture of the quantitative effects involved, a family of similar aeroplanes differing only in size have been considered. These machines are representative of current practice, with spans from 70 to 230 ft., and have the same plan form, mass distribution, wing loading, and strength in terms of ultimate normal acceleration. All are assumed of the same construction utilising the same maximum permissible stresses. These assumptions enable the vibration characteristics of all members of the family to be expressed in terms of span only, and it may be mentioned that only the fundamental mode of wing flexure is of primary importance in this connection. It is reasonable to suppose that the results obtained for this family may be applied without major error to most orthodox present-day designs. Calculations have been made which assume the currently required gradient distance of 100 ft., invariable with altitude, and allowance has been made for the combined effect of aeroplane motion and aerodynamic lag in the buildup of the air loads upon the time history of the applied loads. To a close approximation, this allowance amounts to increasing the gradient distance by 20 per cent and assuming a sine variation of loading with time in place of the linear.

The effect of speed to a first approximation can be found by first obtaining the undamped velocity of vibration and then computing the equivalent damping forces. As the damping forces are proportional to the square of the speed, they increase in relation to the applied force as the speed rises.

Fig. 11 shows a typical plot of overstress factor against speed at sea level, the dotted curve showing the value of the factor if zero damping is assumed. This curve shows clearly the importance of damping at all operating speeds and particularly at the higher speeds.

These curves, unfortunately, show that the maximum overstress factor occurs at around the present-day values of the cruising speed. Similar plots for the whole family at various altitudes have been made and used to construct Fig. 12, which shows the maximum

EFFECT OF FLEXIBILITY ON WING ROOT BENDING STRESS DUE TO PASSAGE THROUGH A GUST.

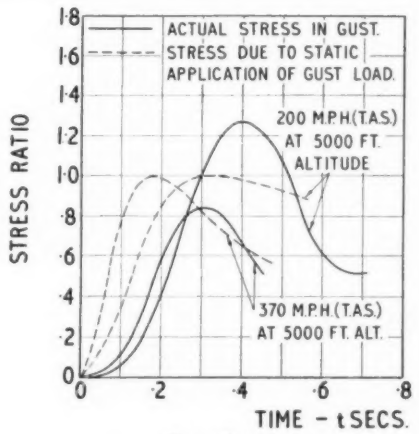


FIG. 10.

VARIATION OF OVERSTRESS FACTOR WITH SPEED
170 FT. SPAN 100 FT. GRADIENT DISTANCE AT SEA LEVEL.

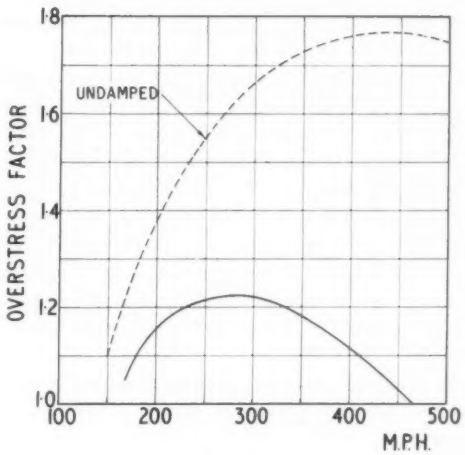


FIG. 11.

overstress factors as a function of span, with altitude as parameter.

The most striking thing about these curves is the rapid curving over at small values of the span, followed by a roughly constant factor for all values of the span from 100 to 250 ft., with, apparently, a drop in favour of very large spans indeed. There is, of course, a simple physical explanation as to why, contrary to earlier expectation, the very large span aeroplanes do not, in fact, experience much greater overstress than the (comparatively) smaller machines. At a given altitude and forward speed, the aerodynamic

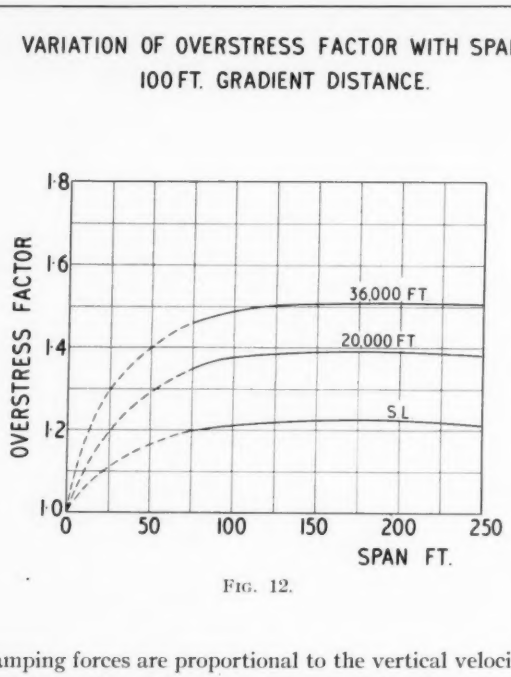


FIG. 12.

damping forces are proportional to the vertical velocity of deformation of the wing. Now, in flying through the same gust, the large machine will experience deflections greater than the small one roughly in proportion to span—i.e., a 250-ft. span machine will experience approximately two and one-half times the deflection of the 100-ft. span. The time in which this deflection takes place, however, is clearly not much greater than for the smaller span, so that the *rate* of deflection and, consequently, the damping force also are about twice that of the smaller aeroplane. Thus it comes about that what the large machine loses due to its slow frequency of oscillation it gains due to superior damping in flexure.

This figure also indicates the effect of altitude. The maximum overstress factor, about 22½ per cent at sea level, has become no less than 50 per cent at an altitude of 36,000 ft. An increase of the gradient distance to 200 ft. in place of 100 ft. would reduce this figure to approximately 30 per cent.

The general conclusions that can be drawn from this study of dynamic response to a gust constant in shape and velocity are as follows:

(1) At any given altitude, the overswing factor increases with increase of speed until aerodynamic damping overrides the gust effect. With further increase of speed, the overswing factor gets smaller until the wings become "dead-beat" or better.

(2) The maximum value of overswing factor is apparently independent of the size of aeroplane.

(3) The speed at which the overswing factor reaches its maximum value decreases with increase of size of aeroplane.

(4) The overswing factors increase with increase of altitude due to decreased damping.

As the true magnitude of these dynamic effects is controlled by the rate at which a gust builds up, it is most important that we should learn more about actual gust structures. If, for example, it could be established that the gradient is proportional to the square of the true velocity of the gust, overswing would be independent of altitude.

In any case it appears that for high cruising speeds the gust cases are actually less severe on large aeroplanes than on small ones, and on this count there is no limit to size.

Gust Alleviation

The detection of gusts at long range by radar depends on cloud formation. While this system appeared promising at one time, recent flight experience has shown that severe disturbances occur in clear air and that long-range gust warning is not therefore possible. We must assume therefore that the aircraft cannot alter course to avoid a gust or reduce speed before encountering it. In certain circumstances it may therefore be desirable to find means whereby the applied loads on the wings in gusts can be reduced so as to permit a reduction of structure weight. There are two methods of achieving this. One is by so arranging the response of the aeroplane that alleviation takes place automatically; the other is by detecting and measuring some quantity associated with the gust (e.g., gust velocity) and using it to operate a servomechanism that supplies the alleviation.

An example of the first type of alleviation is to cause the wing to twist nose down as the load due to an up-gust builds up, thus reducing incidence at the wing tip. This entails either a forward distribution of the flexural and inertia axes or a skew hinge at the root. The flutter and structural problems of these solutions for straight wings lead one to seek more feasible alternatives.

As an example of the second type of alleviation, a symmetric up-gust, which is usually the most severe case, might be alleviated by nose-down pitching of the whole aircraft. Analysis, however, shows that the angular accelerations are greater than normal elevator power can provide. While this approach might be made to work and produce some favourable effect on the structure, it would no doubt be balanced by equal but opposite reactions of the passengers.

By far the most attractive solution makes use of symmetric motion of the ailerons, either by direct mechanical coupling or by a servomechanism. This system, while potentially capable of reducing the bending moment at the wing root by 50 per cent and of almost completely eliminating the bending on the outer wing, does not involve unacceptable large pitching or vertical motion of the aeroplane.

Many types of gust detection and measurement have been suggested. One proposal makes use of wing strain.

The difference in extension between a free rod and the tension surface of the wing across the span is used to operate the ailerons.[16] Two methods of operating the ailerons are possible; in one, the rod operates the ailerons directly. The system of using wing strain is mechanically simple but suffers from appreciable time lags. The peak dynamic response of the wing occurs after the maximum gust velocity is reached. This response activates the ailerons whose effect is subject to Wagner lag, and there is a still further delay before the change in lift produces the dynamic response in the alleviating sense. This means that in practice the aileron is out of phase with the gust by 0.1 sec. or longer for a large aeroplane. This greatly reduces the effectiveness of alleviation, and the implications as regards flutter must be closely watched.

As an example of the flutter implications, consider the wing vibrating in its flexural fundamental mode. When the tip reaches its extreme "up" position, the strain between the tension surface and the free rod will demand an "up" aileron angle. Owing to the lags in the system, this lift due to aileron will not act until the wing is on its downward beat; thus, the downward lift due to aileron acts as negative flexural damping and may be sufficient to reduce the total damping to zero.

The vertical acceleration of the aircraft is a good measure of the gust velocity, but alleviation systems based on this quantity suffer from even greater lags than those using wing strain and must be rejected on that account.

A device ahead of the wing which measures directly the change of incidence due to a gust is probably the most promising method, operation of the ailerons being by some power or servo system. The essential need is to keep the time lags for the set of operations within the small margins permissible so that the alleviation is synchronised with the arrival of the gust. In a large aeroplane by virtue of a long body extending forward of the wings some advanced warning is possible; in the Brabazon I, a distance 80 ft. forward gives a time of 0.2 sec. between the arrival of the gust at the nose and at the wing. Also, the dynamic response provides some smoothing and permits an additional lag of the order of 15 per cent of the wing fundamental period.

Even so, with large power-operated controls, these times still leave the problem a difficult one; the efficiency of alleviation falls off rapidly with increasing lags. Fig. 13 shows the distribution of lags in the response to the British variety of gust for the Brabazon I, the total lag being just acceptable.

A device of this nature needs to be automatically compensated to deal with changes of incidence due to changes of flight speed. As static and manoeuvre margins can also be adversely affected, the system should be insensitive to minor disturbances.

Another precaution that must be observed is in the placing of the measuring meter, for, if it has an amplitude in any mode of the same sign as the amplitude at mid-aileron, the damping in that mode will be reduced. Unlike the previous case of alleviation from wing strain, this effect is not primarily due to the lags in the system. If we consider the case of a detector placed just forward of the aileron, when the wing is passing through its zero position in the upward direction, the device will detect a fictitious down-gust and demand "down" movement of the ailerons. This will cause a lift on the wing acting in the sense of negative flexural damping. The effect is most marked in the case of the fundamental mode, where the total aerodynamic damping can be reduced by as much as 50 per cent; this could easily produce flutter at low speed.

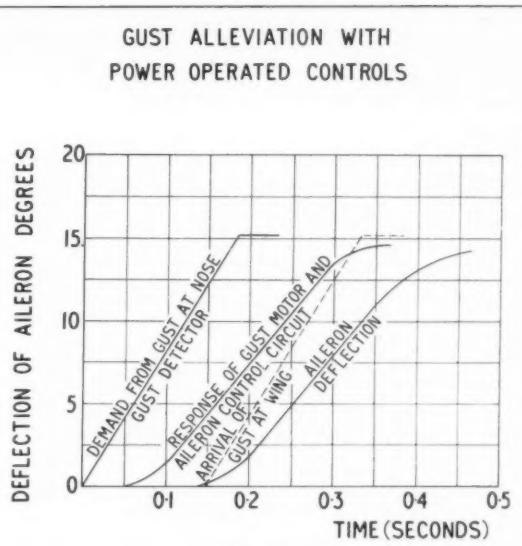


FIG. 13.

The effect can be eliminated by placing the detector at a position where the amplitude of the lower modes relative to the mid-aileron are small and negative. The effect on the higher modes can be neglected.

This elementary discussion of the lag and stability characteristics of gust alleviating systems is intended only to make clear some of the physical factors involved. The system as a whole forms a servo loop; the various contributory lags, both aerodynamic and mechanical, must be separately analyzed and the stability of the whole servo loop studied for a suitable range of frequencies before any particular system can be claimed to be safe. In general, however, the systems with low lags and small feedback will present the less severe problems and are therefore to be preferred.

Flutter

The problem of flutter is one of the earliest associated with flying but has, until comparatively recent times, been solved merely by solving the problem of strength coupled with the tactical distribution of lead weights. We are now becoming proficient at solving the problem of strength and are disturbed if our test structures

do not fail within 1 or 2 per cent of the design loads (however arbitrary these loads may be). At the same time, this steady refinement of design has resulted in a reduction of structure weight for given design loads. Refinement of structural design has reduced the stiffness of the structure, and this, coupled with a steady increase of cruising speeds, has brought the flutter problem into its present prominence.

Flutter is by no means confined to large aeroplanes, although at one time it was considered that flutter of such aeroplanes would involve considerably more detailed analysis than for small aeroplanes. However, it now appears that it is not so much the size of the aeroplane which matters but the type of structure employed and, in fact, that any new aeroplane designed to modern strength theories will require a complete flutter analysis.

Any part of the aircraft structure where a structural deflection results in a change of aerodynamic load on the part is liable to flutter. Thus arise the familiar problems of wing flexure-torsion, fuselage-tail plane, and fuselage-fin flutter. Added to these we have further degrees of freedom due to the motion of a dozen separate analyses which may have to be undertaken before the aeroplane flies—in fact, some considerable time before the aeroplane flies. If the aeroplane is likely to be designed by the flutter requirements, these analyses must all be completed in the early stages of the design.

The criteria available are of limited scope; the wing flexure torsion flutter criterion[17, 18] and the spring and servo tab criterion[19, 20] are examples and cannot be applied to unusual weight or structure distribution. Since most new aeroplanes are unusual in one or other of these respects, the criteria cannot be taken as more than rough guides. Thus the prospect of having to design an aeroplane on the basis of flutter calculations is alarming.

It seems improbable that criteria, however many parameters they include, can be developed to cover the multitude of inertia and elastic systems present in different types of aeroplanes. The only answer seems a complete calculation for each variation of mass, stiffness, or geometry. The major difficulty to be tackled in this field seems, therefore, to be excessive time required for each flutter calculation.

Another approach to this problem can be made by introducing the flutter model either in a wind tunnel or in free flight. That there are many difficulties is immediately apparent when one considers the task of producing a model that not only has the requisite elastic and inertia properties but also must have the correct aerodynamic properties. Because of scale effects it is likely that the model would of necessity be large, ruling out the wind tunnel for all except local flutter effects. Flutter model testing may therefore be limited to those cases where the aerodynamic derivatives cannot be determined with sufficient accuracy for calculations. This is not to suggest that the experimental approach is useless—it is essential that our aerodynamic derivative theories be checked by direct experiment—but that this approach is unsuitable for ad hoc testing of particular aeroplane designs and is best suited to the verification of the basic flutter data.

There remain two (very similar) approaches that we may make to this problem and which both show promise. These are the electric analoge and the electric computing machine.

The electric analoge is a development of the well-known analogy between the equations determining the oscillation of a structure and the equations of the oscillation of an electric circuit. In this, the inductance of the circuit replaces the inertia, the resistance replaces the damping, and the capacity replaces the elasticity of both structure and air. Thus a circuit may be built up which will oscillate at a certain frequency defining an equivalent flutter speed and mode. Once constructed to simulate any particular type of flutter problem, the answer would be given immediately and the effect of variation of structural and aerodynamic parameters easily found.

The electronic computing machine is a device for applying a specified programme of operations to a given set of parameters in a short time. Thus the normal methods of flutter calculations are amenable to treatment by this machine, as, in fact, are all other calculations that require a definite sequence of computations. Once again, the short time required for each complete calculation enables a greater variety of structural and aerodynamic parameters to be considered in the time available.

Both devices suffer from a high initial cost, and it is in this respect that the electronic computing machine, with the advantage of its greater adaptability to all types of design calculation, may be effectively cheaper to run.

Wing Flexure-Torsion Flutter

Although it is impossible to obtain a really accurate value for the flutter speed for any general analysis relying on a criterion and fairly broad assumptions, it is often instructive to observe the general trend as various structural and aerodynamic parameters are altered. The wing flexure-torsion flutter criterion[18] (i.e., rigid aileron) is a reasonable guide to the flutter speed and, although it neglects the effect of variations of mass distribution (e.g., wing engines) and stiffness distribution between different aeroplanes, may be assumed true for this general investigation.

The criterion may be stated as

$$V_{FC} = K_1 (1 - M^2)^{1/4} (m\theta/s_{c_m}^2) \qquad (1)$$

where M is the Mach number corresponding to the flutter speed, V_{FC}, and where a minimum value of $(1 - M^2)^{1/4} = 0.775$ is applied for all values of $M > 0.8$.

K_1 is a function of wing loading, taper ratio, and inertia axis position, although the variation with wing loading is extremely small and may be neglected.

It is now required to determine the effect on $m\theta$ and, hence, V_{FC} of various parameters such as wing loading (w) aspect ratio (A), spar box dimensions, etc., for typical wing structures.

By assuming a certain inertia factor (n) and inertia relief factor (n_r), it is possible to design a wing structure and obtain general expressions for skin and spar web thicknesses in terms of the various parameters. From this thickness distribution a value of m may be determined. This analysis results in the following expression for V_{FC}:

$$V_{FC} = 10^3 K_1 (1 - M^2)^{1/4} \times \sqrt{\frac{nw\,(1 - n_r)}{K_2 \frac{fA}{k_2^2} + K_3 \frac{fc}{k_3}\left(1 + \frac{A_s}{bt}\right)}} \qquad (2)$$

where K_2 and K_3 are constants being about 0.002 and 0.008, respectively, while K_1 is about 65 (where ft.lb. sec. units are employed).

An idea may be obtained from this expression of the effect of various structural parameters on the flutter speed. The effect of aspect ratio (A) and of wing loading (w) is shown in Fig. 14, where it is evident that, although increase of aspect ratio reduces the flutter speed, this may be offset by an increase of wing loading. Thus it appears that, if high cruising speeds are desired for aeroplanes of the future, an increase of wing loading or a reduction of aspect ratio will be inevitable unless the criterion of wing design is allowed to change from strength to stiffness. It must be remembered that a wing designed for strength is inefficient as a torsion member, the skin thickness decreasing from root to tip as the first or second power of the span, whereas on a wing designed for torsional stiffness only the thickness should remain constant or increase if the wing tapers towards the tip. If the stiffness becomes a design condition, it is therefore likely to modify our ideas of the optimum form of compression structure for the wing box and will probably result in a smaller ratio of stringer area (As) to skin area (bt) than has hitherto been considered desirable, this ratio being progressively reduced towards the wing tip. Fig. 15 shows the effect of As/bt and of design compressive stress (fc) on the flutter speed, assuming, however, that As/bt is constant along the span.

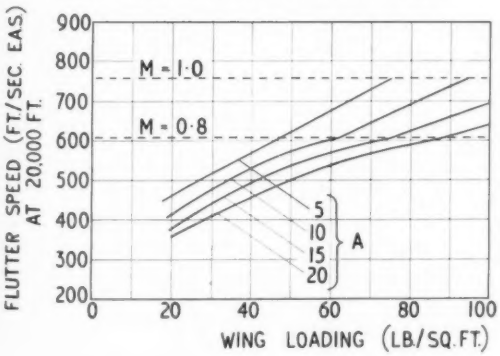


FIG. 14.

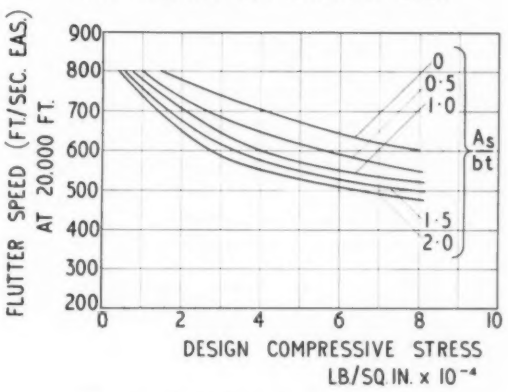


FIG. 15.

A method often proposed for increasing the torsional stiffness and the flutter speed of a wing is to increase the interspar chord (k_2). This generally has an unfavourable effect on the bending efficiency of the box, but the effect on flutter speed is marked, as may be seen from Fig. 16.

It is of interest to note here that the other major aeroelastic effect that fixes the torsional stiffness—i.e., aileron reversal—results in an equation for reversal speed similar to Eq. (2) above but with different values for the constants K_1, K_2, and K_3. The aileron reversal speed as a function of wing loading and aspect ratio is plotted on Fig. 17 and in all cases is higher than the flutter.

The above analysis indicates therefore that the adoption of wing loading of about 80 lbs. per sq.ft. as a method of reducing wing structure weight (based purely on bending considerations) is favourable from the aeroelastic point of view and may allow some increase of aspect ratio (provided, of course, this does

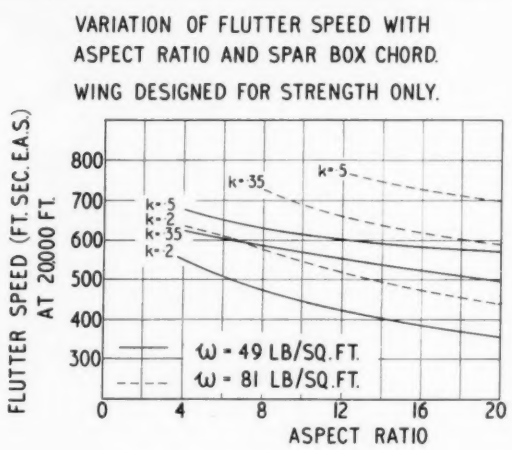


FIG. 16.

not seriously increase the structure weight required for bending strength). However, as has been mentioned previously, no account has been taken of the effect of engine and other masses on the flutter speeds. It is impossible to consider these effects in detail in a general analysis such as this, although it may be said that engines placed forward of the flexural axis usually result in a raising of the flutter speed. There are possibilities here of reducing the required torsional stiffness by a favourable distribution of masses (e.g., engines placed at the wing tip), but much more work is required before such an effect may be included with confidence in the torsional stiffness criterion.

Wing Analysis

The following features that have a bearing on wing design have been considered: (1) the effect of a variation of wing loading and aspect ratio on the performance of an aeroplane and on the weight of fuel required; (2) an examination of the dynamic response of an aeroplane in the gust case and the consequent effect on the stresses and the structure; and (3) the provision of structural stiffness necessary to avoid wing flutter and aileron reversal.

The task now is to determine the layout of a wing that will satisfy airworthiness requirements and permit the highest possible efficiency in operation. This involves the selection of wing loading and aspect ratio involving a balance between the weight of the structure satisfying the demands of strength and stiffness and the weight of the fuel appropriate to the particular performance characteristics. For this purpose some rule must be established whereby the effect of these variables on structure weight can be assessed. The influence of size must also be included. Those who in the past based their forecasts on the square cube law and established a limit to the maximum possible size of aeroplane must by now have diminished confidence. On the other hand, those who entirely ignore the effect or rely on some mystic solution can introduce large errors into their overall evaluation.

Methods of weight analysis fall into four separate categories successively approaching an increased order of accuracy. These stages are appropriate to:

(1) A general survey for the purpose of suggesting probable trends and comparing the relative merits of different forms of layout and types of power plant.

(2) A particular design study of an aeroplane whose form and type of power plant have already been decided as a result of a general survey. In this case the designer is using as a basis an approximate specification of user requirements, and his purpose is to determine the probable gross weight and the optimum combination of span, wing loading, and other characteristics so as to obtain the best possible performance and economy of operation.

(3) In the detail design stage when weight estimates can be made from production drawings.

(4) In the manufactured state when actual weighings can be made of details, components, and the complete aeroplane.

For the first two stages one is forced to rely on a semiempirical cum statistical analysis, taking the results of known aeroplane to establish general coefficients. Care is needed in the selection of the data for this purpose, since each aeroplane is subject to many variables and compromises, often known only to the designers so it is important to separate the various controlling factors. It is likely that one would find a variation

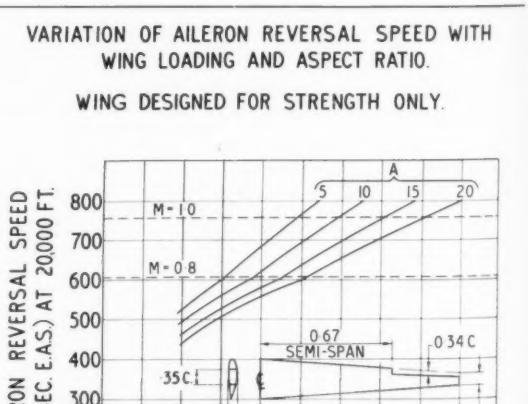


FIG. 17.

in the general terms of formula used for weight estimation as derived by different technicians, but no doubt each design team feels that its own methods have special merit. In novel circumstances the first estimate may be an inspired guess, but this should be followed at the first opportunity by a more fundamental approach.

It is not my intention to describe in detail any particular form of estimate, but, so that an argument can be pursued, it is desirable to outline the effects of relevant factors on wing structure weight.

A wing can be split up into different parts each fulfilling a main function—viz., the compression structure and ribs, the tension structure, the shear members, leading and trailing edges, flaps, and controls.

Taking the compression structure and ribs, it has been demonstrated that the instability failing stress f of a typical panel is given by the equation

$$f = K\sqrt{PE/l}$$

where

P = compressive end load per in. width
l = panel pin centre length
K = a constant depending on the geometry of the stiffeners

The value of K is a measure of the radius of gyration per inch width of the panel and can be used to compare the relative efficiencies of different forms of stiffener.

This formula gives a good indication of the layout of wing structure that should be adopted to obtain minimum weight. At a chordwise section at or near the wing root, the instability failing stress f should be equal to the yield stress of the best material available. If a form of wing skin stiffening having a high value of K is selected, the desired stress can be achieved by arranging the wing geometry to give the required ratio of P/l to satisfy the equation. That is to say that, by a method of successive approximation the width and depth of the main structural box at the root is made to control the value of the compressive end load per inch resulting from the spanwise applied bending moment and the rib pitch fixes the panel pin centre length. Clearly, this optimum ratio of P/l should be maintained as far as possible outwards along the span. To assist in this aim the wing should have a straight taper. Also, resistance to the applied bending moments should be confined to a main structural box, and contributions from other secondary portions such as the leading and trailing edges should be kept as small as possible. This could be achieved by providing a nonstrain-carrying connection between the primary and secondary structures. The structure of the secondary portions can then be made as light as possible, since the only forces it has to carry arise from the air loads on its own cover. These are transmitted directly to the main box. This is a conception that in practice is difficult to attain completely.

The weight of the sum of the ribs and the compression structure can now be estimated. Away from the wing root, the rib pitch plays an important part in maintaining an optimum value of the ratio P/l.

If the weight of a typical rib is expressed in the form dt_r where d = depth of rib, t_r = equivalent thickness of rib, and ρ = density of material—the weight per unit area of a typical portion of the compression structure and ribs can be simplified into the form

$$W_1^2 = \rho\,[(dt_r/l) + (1/K)\sqrt{Pl/E}]$$

The rib pitch can now be varied to make this weight a minimum by equating $\partial w_c/\partial l$ to zero so that

$$W_c^1\ (\text{min.}) = \rho\,(dt_{k_r}P/E)^{1/3}\quad [3/(4k^2)]^{1/3}$$

P, the end load per in. width, can be expressed in terms of the load factor, wing load, wing span, and wing depth, so that

$$W_c/W \propto (Nb^2\,t_r/\omega 2)^{1/3}\,(1/K^{2/3})$$

It is now possible to see what saving can be had by an improvement in skin-stiffener design. The value of K appropriate to a well-designed Zee section is 0.8, whereas it is possible to achieve a value of 1.1 if a combination of inverted Y sections and minor intermediates is used. It is probably optimistic to assume that this arrangement could be maintained all the way to the tip, since the section thicknesses would be limited by manufacturing processes; therefore, a mean value for K of 1.0 should be used. Inserting these values in the term containing K shows that the weight of the ribs and compression structure could be reduced by about 13 per cent by a change from Z stiffeners to the more efficient combination.

By comparing the component parts of suitable large aeroplanes, a coefficient can be found whereby weight estimates for other aeroplanes can be made. The part of the equation defining the compression structure plus rib weight is then

$$W_c/W = 43\,\rho\,(N_b^2 t_r/Ew^2)^{1/3}\,(1/K^{2/3})$$

The remaining parts of the wing—i.e., the tension, shear, and secondary structures—have more direct relationships with the load factor and wing geometry. Their weights can be expressed in a more simple form, and a series of coefficients can be found by equating them to the known weights of the similar components of existing aircraft.

With a basis for estimating the weight of the structure for different sizes of wings subjected to specified applied loads and taking into account the dynamic effects previously discussed, it is possible to draw up a general picture from which the weight of a particular wing can be assessed.

The gust cases will usually provide the design strength criteria for the wings; typical load factors for various

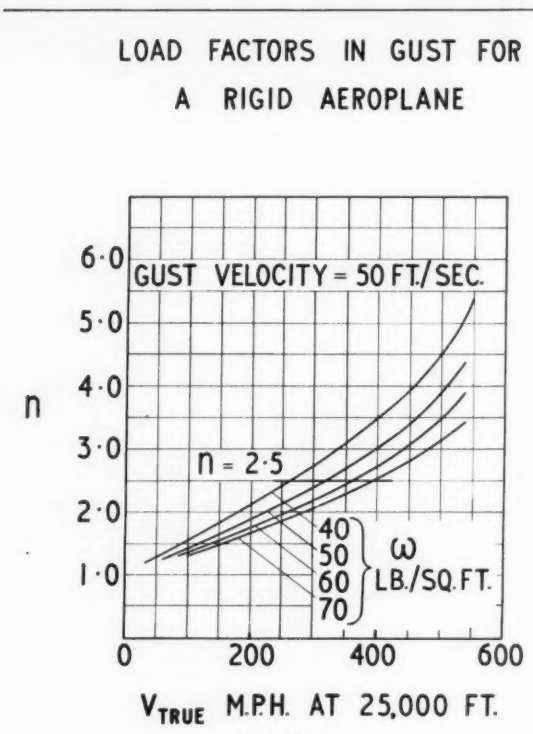


FIG. 18.

wing loadings and cruising speeds are shown in Fig. 18. These curves apply at altitudes below 25,000 ft., since some relaxation in the gust indicated velocity is permitted at greater altitudes. As the performance of aeroplanes fitted with turboprop engines is such that the maximum indicated cruising speed increases with decreasing altitude, to avoid excessive load factors it may be necessary to limit the maximum permissible indicated speed below some specified altitude. This should not be a handicap in normal operation.

The factors shown in Fig. 18 should not be used in the formula for weight estimates without making an allowance for overswing. Following the discussion where it was shown that, while a constant overswing stress factor of 1.2 may be taken at a speed of 375 m.p.h. at ground level for any wing span between 100 and 250 ft., the factor increases with increasing altitude, the reason being the reduced effect of aerodynamic damping. While this may be so, it is unlikely that such unfavourable conditions would be accepted in the design of an aeroplane when alleviation is possible by the introduction of some suitable device.

With these assumptions the wing structure weights shown in Fig. 19 have been produced. Two sets of curves are shown: the one for which the ultimate strength factor is 3.75 and the other for the appropriate gust factor at a true speed of 375 m.p.h. at 16,000 ft., with a representative overswing factor of 1.2, and a factor of safety of 1.5. The difference between these two sets of curves shows the effective saving to be had by efficient gust alleviation.

Obviously, Bristol aeroplanes fall into their appropriate place among the curves as the coefficients in their equations were derived from them.

Optimum Wing Loading and Aspect Ratio

At this stage it is convenient to combine the results of Fig. 19 with those of Tables 3 and 4 to form an opinion of the influence of wing loading and aspect ratio on operating economy (see Table 5).

TABLE 5

Wing Loading	Aspect Ratio	Fuel for 5,000 Miles Range		Wing Structure Weight	Fuel + Wing Weight
45	10	0.333 W		0.132 W	0.465 W
55	10	0.348		0.123	0.471
65	10	0.362	(a)	0.117	0.479
75	10	0.376		0.112	0.488
45	10	0.333		0.160	0.493
55	10	0.348		0.140	0.488
65	10	0.362	(b)	0.125	0.487
75	10	0.376		0.115	0.491
55	8	0.388		0.111	0.499
55	10	0.348	(a)	0.123	0.471
55	12	0.309		0.135	0.444

(a) With gust alleviation
(b) Without gust alleviation.

The following conclusions can be drawn:

(1) With no gust alleviation and with an aspect ratio of 10, the combined weight of fuel and wing structure weight is sensibly constant over a wide range of wing loadings, and, hence, there is no optimum value to give maximum pay load. This obviously encourages the adoption of as high a wing loading as possible so as

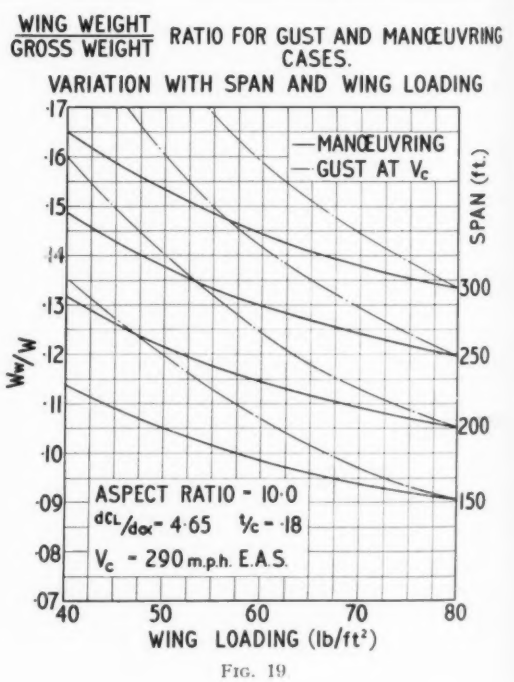


FIG. 19

to produce the smallest possible size and the highest possible cruising speed. An upper limit may be set by any of the following reasons:

(a) A minimum acceptable altitude after an engine has failed so as to be able to fly above rough weather. This is only likely to be critical in the tropics.

(b) The need to be able to take-off from short runways at airports situated in the tropics and at considerable altitude.

(c) A volume limitation to contain the fuel. With jet pipes passing through the wing, landing-gear stowage space, heated air passages in the leading edge for anti-icing, and the rear portion occupied by flaps and their operating mechanisms, the wing area can, in the case of long-range aircraft, be determined by the fuel capacity.

(2) When gust alleviation is incorporated, a greater pay load can be carried if a moderate wing loading is selected. The difference between 55 lb. per sq.ft. and 75 lb. per sq.ft. wing loading is equivalent to a difference of pay load equal to nearly 2 per cent of the gross weight.

(3) The aspect ratio should be as high as possible. Of the aerodynamic variables, this characteristic shows the highest profit. With a high aspect ratio the aeroelastic criteria are more easily met if the wing loading is also high. Even at moderate wing loadings there seems to be no reason why values in excess of 12 or even 15 should not be adopted.

Remainder of Structure

Fuselage.—An analysis of the effects of different forms of fuselage is complicated by many interrelated operating conditions. If an aeroplane is designed to operate on one specific route, only then can the fuselage size and passenger accommodation be arranged to provide facilities just in accordance with the available capacity pay load.

If a long-range aeroplane is required to carry greater pay loads at shorter distances, then its efficiency at long range may be seriously impaired.

Suppose, for example, that a civil transport aeroplane with a range of about 5,500 miles has the most simple form of single-deck circular fuselage, having a floor area just sufficient to accommodate the appropriate number of passengers. Then, if a considerably greater number of passengers must be carried at lesser ranges, a double-deck or "double-bubble" fuselage must be provided. Assuming that the gross weight remains the same, the following changes take place: (1) a small increase of drag due to greater surface area; (2) the greater surface and floor areas, together with the additional windows and doors, increase the structure weight; (3) similarly, the weight of furnishings and soundproofing are increased; (4) heavier air-conditioning plant and ducting; (5) heavier electrical installations.

The sum of these changes may raise the weight of the complete furnished and equipped body from about 14 to 17–18 per cent of the gross weight. Assuming, too, that the maximum pay load is increased from 9 per cent to 15 per cent, the total increase of loaded body weight is about 10 per cent. Now at constant gross weight there must be an equivalent reduction of fuel weight, thus reducing the relieving loads on the wing and thereby increasing the bending moments. The consequent increase of the wing structure weight is about 1 per cent. The landing weight is also greater, so that those parts designed by landing loads will also suffer.

On this reckoning, if this double-deck fuselage is adopted for a long-range aeroplane, the basic operational weight may be 4–5 per cent greater than would be the case for a circular fuselage. After making allowances for the remaining structural parts, items of equipment, crew, and passenger services, the two arrangements are compared in Table 6. The fuel for range is taken from Fig. 4.

TABLE 6

Comparison of Single- and Double-Deck Fuselage Nett Pay Load-Range

		Single Deck	Double Deck
Volumetric pay load		9%	15%
5,500-mile range	Nett pay load	9%	4.5%
4,800-mile range		9%	9.0%
3,900-mile range		9%	15.0%

In this particular example, at a fixed all-up weight of aeroplane, the arrangement, with the body designed specifically to match the pay load at 5,500-mile range, shows at this range an advantage in the ratio of 2:1 over the arrangement with the body designed to match the pay load available at 3,900-mile range. At this lesser range the advantage is reversed.

The only general conclusion I wish to draw from this example is that a penalty is inevitable if one tries to compromise long-range utilisation by excess capacity appropriate to lesser range.

When the principal application for which an aeroplane is intended permits a greater load to be carried, then the double deck may be the most economic arrangement.

Runways and Landing Gear

It has frequently been asserted that for large landplanes the cost and constructional problems in providing runways of adequate length and strength would become prohibitive. This need not in fact be the case, since it can be shown that multiwheel landing gear will more than compensate the effect of the increased weight upon the runway by suitably spreading the load.

Several methods of spreading the load upon the runway are open to selection. The aeroplane may have a

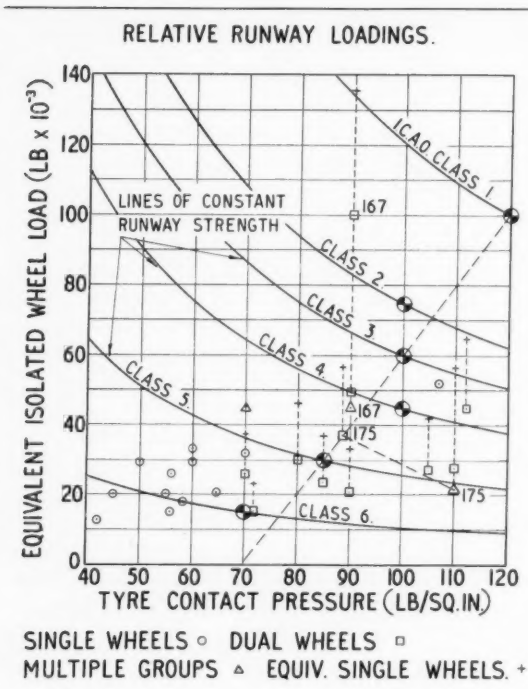


FIG. 20.

number of widely spaced main units each carrying one or two tyres, or it may have few main units with groups of wheels. Track units are a third choice but are fundamentally unsuited to the prepared runways under consideration.

Before the merits of various forms of landing gear are assessed, it will be necessary to consider the general problem of runway bearing capacity. Runways and perimeter tracks can be classified broadly as "flexible pavements" and "rigid pavements." In the former group the subsoil is compacted and covered with a surface of tarmacadam, asphalt, or similar material that protects the subsoil from moisture and from break-up in dry weather, and, if the subsoil is properly graded, this construction can withstand considerable contact pressures without appreciable settlement. The "rigid pavements" are essentially beam systems capable of distributing wheel loads over an area appreciably larger than the tyre contact area. Concrete or reinforced concrete slabs upon consolidated subgrade are in this group.

"Flexible pavements" suffer damage by indentation reaching a permanent critical deflection. "Rigid pavements" characteristically suffer slab corner breakages and sometimes transverse failures due to successive hoggings and saggings under rolling loads. Failures are aggravated by water penetration and erosion or differential settlement of the subsoil. The characteristic behaviour of the two different groups of pavements has led to differing approaches for the assessment of the loading effects on runways. The one is related to the effect of a load upon the deflection of a continuous flexible pavement, and the other is dependent upon the stresses in the corner of a discontinuous slab pavement laid upon prepared subsoil.

From theory and experiments in soil mechanics, numerous workers have shown that the capacity of a runway structure to sustain a load W depends on the contact pressure p and that $W^a p^b$ is a constant. Experimental values for the indices, based upon tests on both "rigid" and "flexible" pavements, have provided the relationship of

$$W^{0.56}\, p^{0.44} = \text{constant}$$

This agrees closely with the theoretical conclusions of Rodin's deflection theory,[21] which gives

$$W^{0.5}\, p^{0.5} = \text{a constant}$$

As a pair of wheels and tyres are moved apart, the carrying capacity of the runway as limited by the critical deflection increases, since, by the method of superposition, the deflection beneath one wheel due to both can be shown to diminish. Average dual wheels as now in use are spaced about two tyre widths apart and permit an increase of about 50 per cent over the load carried on a single wheel with the same tyre pressure.[21] When the dual wheels are about 4 tyre widths apart, the gain is 75 per cent and is about as great a gain as is economical with dual wheels.

If wheels are spaced in tandem pairs instead of side-by-side, the reduction in severity of runway loading is comparable with that obtained by side by side. The use of larger numbers of wheels with tyre contacts spaced both laterally and fore and aft will in most cases give a satisfactory reduction in runway loading. A suitable arrangement is the four-wheeled bogie, which can be about two and one-half times less severe on the runway than an alternative single wheel. This saving and most of the foregoing points are illustrated by Fig. 20. In this diagram the curved lines give constant runway loading and are drawn for the strengths of runways defined by the ICAO Runway Strength Classes, which are written in terms of unique combinations of wheel loads and tyre pressures.[22] The reduction in tyre contact pressure necessary for the carriage of very heavy loads on a single wheel is seen to be prohibitive. The benefit derived from the use of dual wheels is shown by the reduction from single-wheel load to "equivalent single-wheel load" shown plotted at the same contact pressure for typical aircraft as Nos. 13, 14, 15, 16, and 17. The load on one main wheel group of the Brabazon I is about 135,000 lbs. This is reduced to an equivalent of 100,000 lbs. on the runway by the use of dual wheels with tyres at 20-in. centres.

When the four-wheeled eight-tyred bogie, which replaces the dual twin-tyred wheels, is fitted, the equivalent effect is reduced to that of a load of only 50,000 lbs. at a contact pressure of 90 lbs. per sq.in. In a similar way a smaller aeroplane, having a static

load per wheel group of about 56,000 lbs., has an equivalent single-wheel load of about 36,000 lbs. at 90 lbs. per sq.in. with dual wheels (side by side) and an equivalent single-wheel load of only 22,000 lbs. when a four-wheeled bogie is employed at a contact pressure of 110 lbs. per sq.in.

The lines of constant runway strength enable ready comparisons to be drawn. It will be seen that the four-wheeled bogie at 110 lb. per sq.in. tyre pressure requires a lower ICAO class of runway than the same aircraft with dual wheels at 90 lbs. per sq.in.

In the development of Rodin's deflection theory of failure in reference 21, the natural outcome was to use *Cwp* as a Runway Loading Index, where *C* is a correction factor for multiple wheels and tyre contact shapes, and on this basis an aircraft can be compared with another aircraft or with a runway strength even when the contact pressure is different. The ICAO strength classes have equivalent indices as follow in Table 7.

TABLE 7

Class	Bearing Capacity for Isolated Wheel Load of (Lbs.)	Tyre Pressure (Lbs. per Sq.In.)	Load Index (Lb. per In.)
1	100,000	120	3,460
2	75,000	100	2,740
3	60,000	100	2,450
4	45,000	100	2,120
5	30,000	85	1,600
6	15,000	70	1,025
7	5,000	35	394

If large aeroplanes are equipped with load-spreading landing gear, it can be expected that runways greater than ICAO Class 3 strength need not be provided for aeroplanes up to 300,000–400,000-lb. gross weight.

At least two factors conspire to restrict tyre pressures and sizes. High landing and take-off speeds will result in braking and tyre difficulties if the relative size of wheels is reduced as performance and weight increase. Also, a trend towards higher tyre pressure and surface speeds will increase the liability to tyre damage from stones and other objects and will demand an uncomfortably high standard of runway surface cleanliness. This situation is a deterrent in the use of very high pressures, although tyres at inflation pressures of 180 lbs. per sq.in. may be found to be satisfactory after suitable development.

The most suitable form of landing gear for larger aeroplanes is one having groups of wheels. That this trend in development does not require an increase in the proportion of weight is supported by the evidence of Fig. 21, in which the percentage of the gross weight devoted to landing gear has been plotted for some aeroplanes. It can be observed that no increase in percentage weight need be expected with growth in size.

One further aspect of the landing-gear arrangement deserves comment. Having demonstrated the desirability of load-spreading wheel groups, then location relative to the aeroplane structure is of some importance, since in large aeroplanes with relatively flexible wings the dynamic effects of landing are considerably modified by the conditions of support. It is desirable to place the wheel group on each side at approximately the radius of gyration in roll. The nett result of this arrangement has proved lighter than alternative arrangements of three and four groups of wheels without placing great demands on runway width and strength.

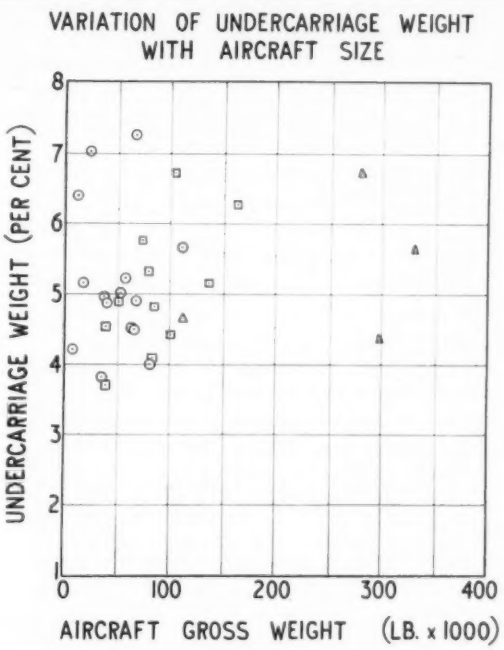


FIG. 21.

CONCLUSIONS

The results of this analysis of transport aeroplanes designed specifically for long-range operation and fitted with turboprop engines show that the wings are the only component of the structure where the proportion of weight inevitably increases with increase of gross weight. The actual amount of increase will depend on the values of wing loading and aspect ratio selected. If these values are maintained constant, the order of increase is approximately $1^1/_2$ per cent of the gross weight for each increase of 100,000 lbs. in gross weight. The minimum basic value for reference should be that appropriate to an aeroplane of 100,000-lb. gross weight.

This means that for an aeroplane of, say, 300,000-lb. gross weight it is possible to design the wings having a weight of 12 per cent and the total structure including the fuselage, tail unit, landing gear, flying control system, engine mountings, and cowlings having a weight equal to 28 per cent of the gross weight.

It has been my purpose to discuss fuel, electric, hydraulic, and other equipment necessary for the functioning of the aeroplane and its engines, not opera-

tional and passenger services. This information can best be analysed by gathering evidence from established aeroplanes, and this procedure is normal practice to all investigators interested in design trends. The relative weights of the various aeroplane services will vary from one installation to another according to local preference, but their sum total does show a relative reduction with increase of gross weight. Also, it is fairly evident that the route to be flown will fix some of the operational equipment and that much of the remainder will not vary at the same rate as the gross weight.

Again, as the proportion of pay load diminishes with increasing range, the weight of air conditioning and other passenger services, being largely dependent on the number of passengers carried, is also relatively smaller.

Thus it is not surprising to find that in practice the ratio of the weight of a fully equipped and furnished structure decreases slowly with increasing gross weight.

To claim that the installed weight of turboprop engines is also less than that for piston engines is true, but a claim on this basis that the percentage basic operational weight is also reduced, thus giving a higher disposable load, is fictitious if used as a comparison with piston-engined aeroplanes. In two particular cases it has been found that the sum of engine plus fuel weight is approximately constant at long range for alternative piston and turboprop engines, thus leaving a constant pay load. However, for long-range operation the turboprop engine gives an advantage of approximately 100 m.p.h. at most economic cruising speed.

In this lecture I have confined myself to certain physical aspects of large transport aeroplanes. While this side is the major reason for the employment of scientists and technicians, it is but the key that unlocks the door to the debating chamber. Within, other important questions must be asked before such aeroplanes can be justified. Certainly, the results of an estimate of economy in operation will be an overriding factor in the determination of policy. An authoritative assessment of this factor is due to Masefield,[23] who, in presenting the Fourth British Commonwealth & Empire Lecture in London, made a classic contribution towards establishing data for use in the evaluation of the economics of civil aviation. Often a careful analysis of all the items in the buildup of fixed annual costs, hourly cruising costs, and take-off and landing costs and making the necessary allowances for fuel reserves appropriate to normal civil operation, he derived the optimum stage length for a series of aeroplanes covering a wide range of gross weight. Conversely, he inferred the most economic size of aeroplane to suit specified stage distances.

On the assumption that in each instance it was possible to achieve 300 hours of annual utilisation with a 65 per cent passenger load factor, he found that for 3,000- and 3,500-mile stage length the optimum gross weight was 400,000 and 400,000 lbs., respectively. Moreover, he showed that the lowest possible operating cost per revenue ton-mile does not start to rise appreciably until the stage length exceeded 3,500 miles.

On a route such as the North Atlantic, where medium-range aircraft are subject to frequent deviations in adverse weather to off-route airports, an aeroplane with longer range will both fly a shorter overall distance and take less time for the journey. Thus for long-stage distances the economic factor is favourable to large aeroplanes.

But in the end the customer is always right, for the psychology of the travelling public does much to determine the passenger load factor when a choice is available.

With increasing size of body, more scope is possible in the layout of seating, lounges, and toilet arrangements, including possibly two or more standards of comfort. One has a simple faith that passengers will prefer to have a little more space provided for their accommodation, and, while all will appreciate the lower noise and vibration levels of turbine engines, the few with an aeronautical upbringing will, in time, cease to speculate on the hazards of explosive decompression.

Summing up the overall picture, when long range is a necessary or desirable requirement for civil air transport, the role can only be filled by large aeroplanes. The point has not yet been reached where the maximum operating efficiency is adversely affected by fundamental technical barriers.

Acknowledgments

I wish to thank the Directors of The Bristol Aeroplane Company, Ltd., for their permission for the publication of the information given and all those members of the staff who have contributed to the methods upon which the data presented are based.

Also I am grateful to a number of my colleagues who have actively assisted in the preparation of this lecture. In particular these include F. C. Marchant and G. M. Lewis, of the Project Department, Engine Division; C. F. Toms, of the Performance Section; and D. Farrar, H. Giddings, H. J. Crampton, N. F. Harpur, and D. I. Vickery, of the Structures Group, Aircraft Division. Their aid has been as unstinting as it has been sustained up to and after the climax of our recent years' efforts.

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Discussions of the Lecture

R. M. Hazen (East Coast), Director of Engineering, Allison Division, General Motors Corporation: The stature of the Wright Brothers Lecture Series has been increased by Mr. Russell's fine contribution. His modest lecture gives an inkling of not only the great ability and effort put on the large transport problem by him and his associates but of the vision and courage required to see it through to completion. Having both the airplane and the turboprop engine under one roof, so to speak, has been unusual and perhaps has contributed to the well-rounded picture he has presented.

From the turboprop engine point of view one can search for important factors that have been missed but can conclude that any discussion must be confined to observations on points already mentioned. It might be observed in a large transport with four dual unit turboprop engines that there are really five altitudes at which the power plants can be running at or near their maximum efficiency by using 4, 5, 6, 7, or 8 operating power sections and that these can be used without disquieting effect on the passengers since all propellers would be turning. While specific fuel consumptions would go up somewhat with reduced expansion ratio at lower altitudes, it would appear that a number of actual operating conditions exist, such as head or tail winds, storm or icing conditions, cabin supercharging limits, standby and descent, which would find this advantageous in maintaining range ability on individual flights.

Our experience indicates an extremely rapid rate of development of power output in turbine power plants of a given size as compared to the reciprocating engine, due primarily to the unexplored possibilities of the former. There is continuous pressure for improved specific fuel consumption on turbine engines. These can be obtained by either or both component efficiency and higher temperature operation in a given size engine, but they result in concurrent power increases of a fairly large order. This may result in the power plant outgrowing an existing transport airplane. Does Mr. Russell consider that an existing airplane can be adapted to such a rate of change within present airworthiness limitations at reasonable costs in development and tooling by increasing operating altitude and/or cruising speeds? Would he consider "shedding" or step-by-step elimination of power sections as a feasible alternate? It is reasonable to assume that a relatively small number of turboprop engine sizes will be developed, tooled for production and available to commercial transport operators at reasonable cost in view of the variety and kinds of propulsive units which the military services can support. The problem of keeping transport aircraft and available engines matched may involve a lot of thought and seriously affect the "optimum" power plant availability too frequently assumed. It is with this consideration that it is my belief that new turboprop transports not now in existence are unlikely to be designed and that the expectation should be that turboprop conversions of developed and tooled current reciprocating powered transport and cargo aircraft will predominate. It would also seem advisable that such airplanes cover military transport and cargo requirements, as well as

commercial, in order to maintain reasonable costs for all concerned.

One other point is desirably clarified with reference to power plant plus fuel weights of turboprop and reciprocating engines. Frequently, best economy fuel-air ratios of 0.058 to 0.062 are used on reciprocating engines when compared to those of current allowable operating temperatures on turboprop engines. It appears to me that most of the reciprocating engines are about as far away from actual satisfactory operation at their optimum cruise condition because of the lead fouling problem on high-octane fuels as turboprop engines are from operation at 150°–200°F. higher turbine operating temperatures. If this is so, the initial weight advantage of the turboprop is available for any range the airplane may fly, and the cost of fuel used maintains approximately a one-third advantage in the turboprop engine. Other factors affecting this question are the oil tank weights and oil consumption weight and cost in favor of the turboprop and the current availability of these engines at well under $^1/_2$ lb. per e.h.p. Would these considerations change the conclusions regarding relative range and pay-load capacity of the turboprop and reciprocating-engine installations?

Dr. N. J. Hoff (East Coast), Professor of Aeronautical Engineering, Polytechnic Institute of Brooklyn: Of the 13 Wright Brothers Lecturers, eleven have been research scientists, professors, and research administrators, and one a test pilot. Mr. Russell is the first designer to be asked to deliver the Wright Brothers Lecture. This is a special reason for us to give him a cordial welcome, because the achievements of Wilbur and Orville Wright in the field of airplane design certainly did not lag behind their accomplishments as research men and pilots. As is well known, Mr. Russell is the designer of the Brabazon I, which flew for the first time last September. In the design of this 300,000-lb., 20,000-hp. transatlantic transport, many difficulties had to be overcome. Those familiar with the work going on at the Bristol Aeroplane Company know that the problems were attacked with the best available methods of experimental and theoretical research and that no effort was spared to get the optimum result. Mr. Russell's paper gives a picture of the most up-to-date approach to the problem of the design of very large transport aircraft. It will undoubtedly be used as source material by those interested in today's state of development of the art.

Perusal of the manuscript shows the tremendous change airplane design has undergone in the last 10 years. Before World War II, airplane stress analysis was based almost entirely upon statics. In the design of the Brabazon, static conditions were replaced to a large extent by the analysis of dynamic loads. Almost half of Mr. Russell's paper deals with problems of aeroelasticity. To save weight, Mr. Russell found himself compelled to install in the Brabazon a gust alleviator, whose purpose is to cut off the peak loads due to dynamic effects. The alleviator permitted a saving in structural weight amounting to approximately 2 per cent of the gross weight of the plane, which meant an increase in pay load of about 5,000 lbs. From the economic standpoint, this increase may make the difference between failure and success in commercial operations, and it is most encouraging that both an airplane and a transport company are willing to take the bold step of trying out such a device. The step is undoubtedly a bold and imaginative one. The installation is complex mechanically and electronically, and many a transport company will be reluctant to use an airplane whose structural integrity depends upon the proper operation of such a complex installation until it has been proved over a number of years. The problem is one of added pay load versus the possibility of structural failure under adverse weather conditions due to improper operation of a complex servo installation. But progress cannot be made without courageous decisions, and a pilot from the time before World War I would consider the mechanical and electronic equipment of our smallest and simplest personal planes as unnecessarily complicated devices compromising the safety of flying. The question arises now whether it would not be more suitable to try out the operation of a gust alleviator on a heavy bomber. The weight saved by the alleviator could be used to provide more protective armament and armor. In wartime, the accruing increase in safety would more than offset the danger of a mechanical or electric breakdown of the alleviator. Of course in peacetime, safety considerations are similar for bomber and transport.

As the diameter of the Brabazon fuselage is about 200 in. and the stringer spacing is small, one might expect the fuselage to fail in general instability when subjected to a static bending test. An approximate analysis, by the method of *N.A.C.A. Technical Note No. 1499*, of data received from Mr. Russell yielded a value of 214 for the buckling index. On first sight one would conclude, therefore, that the frame spacing is too small and that weight was wasted by incorporating too many frames in the fuselage. It turns out, however, that the general instability stress by far exceeds the proportional limit of the material and that the reduced modulus of the stringer decreases considerably the value of the buckling index. The calculations could not be carried out completely because the Brabazon fuselage is outside the range covered by *Technical Note No. 1499*, and the material properties were not known exactly to this discusser. Nevertheless, there is little doubt that the danger of general instability is only slightly greater than that of panel failure. It appears most likely that failure would be initiated by the local instability of an unsupported edge of a stringer.

Mr. Russell and the Bristol Aeroplane Company must be complimented on the amount and excellence of the work that went into designing the compression members of the wing structure. The arrangement having the maximum strength-to-weight ratio was determined analytically in England, while at the same time in America the N.A.C.A. carried out a long series of experiments to the same end. In this connection two remarks may be in place. First, it appears that scientific intelligence between our two countries was not very good, because the two groups did not know of each other's work until both the analysis and the experiments were completed. Second, the difference in the approach to the problem seems to be typical for our countries, in spite of the rapid increase in theoretical research in the United States in the last 20 years. Apparently, we have a little more money for expensive experimentation, while the British perhaps can get highly trained analysts at a somewhat lower cost.

In conclusion, I should like to quote from a lecture delivered by Major J. D. Rennie in order to show how much aviation has developed since the early twenties. The paper was printed under the title "Some Notes on the Design, Construction, and Operation of Flying Boats" in the *Journal of the Royal Aeronautical Society*, Vol. 27, No. 148, p. 123, April, 1923:

> "Over long sea routes, say one hundred miles and upwards, the flying boat possesses overwhelming advantages over its only rival the steamship. . . . . Taking the maximum speed of a ship as 20 knots and the cruising speed of a flying boat as 80 knots, the voyage would be accomplished in a quarter of the time."

Ray D. Kelly (East Coast), Superintendent, Technical Development, United Air Lines: It is most refreshing to find so comprehensive an analysis of the characteristics and design problems of the turboprop transport aircraft as has been given us by Mr. Russell in this Thirteenth Wright Brothers Lecture.

So many proponents of turbine-powered aircraft have jumped so quickly, and apparently so easily, to the turbojet configuration that some of us have been left gasping as to how the practical operating problems are to be mastered. Mr. Russell does not attempt to approach closely to these practical operational problems, but at least he has established broad parameters for all to see and appreciate, so that the day-to-day experiences and requirements of the civil aircraft operators may be more effectively applied to turboprop-powered aircraft.

In general, this lecture indicates that the turboprop-powered commercial transport has great potential possibilities for air-line operations, although a transition to it from our present reciprocating-engine-powered craft represents a step of considerable magnitude from an operations standpoint. The apparent lesser degree of flexibility will not be easily adapted to present operational methods and patterns. The higher rate of fuel consumption, particularly at other than optimum altitudes, is significant. The probable need to design the aircraft to fit a specific route segment and to build, generally, large aircraft (up to 300,000 lbs.) makes the venture into the new field somewhat alarming when viewed from the standpoint of privately financed air lines operating in stiff competition.

However, if the changeover to turboprops appears to be of considerable magnitude, that of going directly to turbojet has many more disturbing elements. To me at least, the turboprop appears to be a logical next step for the medium- to long-range civil air transports, and I am grateful to Mr. Russell for giving us so comprehensive an analysis of the pertinent design criteria.

It is believed that the following good operational characteristics can be claimed for a properly designed turboprop transport and that these are consistent with Mr. Russell's design criteria:

(1) Improved take-off and climb ability, making it possible to operate still faster and heavier transports out of existing airports. (Enlarged landing areas close to population centers are not easily, cheaply, or quickly achieved. Many major airports are marginal for present equipment.)

(2) A reasonable and usable increase in cruising speed of about 100 m.p.h. can probably be achieved without greatly deviating from the aerodynamic characteristics or safety features of present modern aircraft.

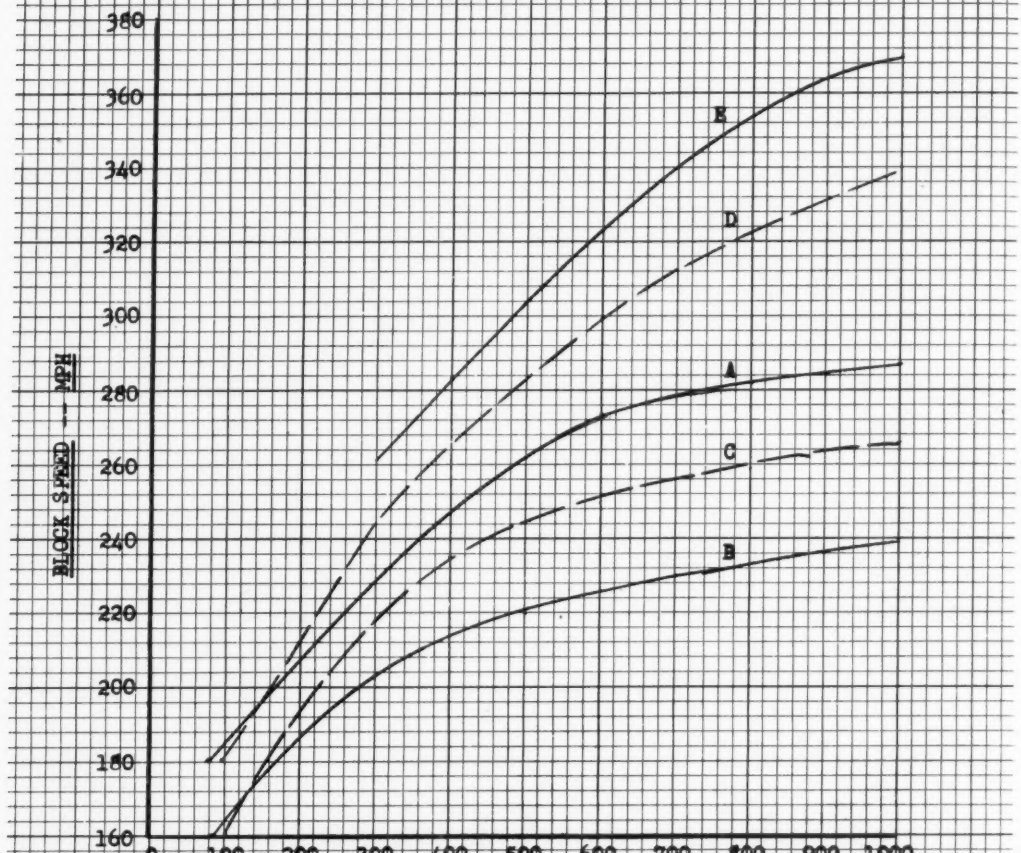
(3) Ground handling characteristics will be essentially equivalent of present piston-engined craft. The important feature of propeller braking will be retained.

(4) The economic considerations for turboprops should not be too greatly different from those of present reciprocating engines. Even now, it appears that the fuel costs of the turboprops may not be too much greater on a ton-mile basis. Mr. Russell seems to think that this is true. If this is the case, then there seems to be every reason to believe that the later improvements in turbine efficiencies will possibly give specific fuel consumptions that are better than that for the current reciprocating engines. (This is extremely important to the air lines, because, even at the present time, some 10 to 12 cents out of every revenue dollar must be immediately paid out to the fuel producers for aviation gasoline.)

It is encouraging to note Mr. Russell's comments that passenger comfort with relatively large airplanes and at speeds in the range of 400 to 500 m.p.h. is likely to be improved. The question of passenger comfort at high speeds is one that is asked by all riders, inasmuch as most of them have had some unhappy rough-air experiences in present equipment. Even some of the information which has been received from military sources has surmised that it may be un-

FIGURE 1

BLOCK SPEED COMPARISON



AIRPLANE	TAXI & RUN-UP	CRUISE BHP/ENG.	TOTAL CRUISE BHP	CRUISE SPEED	CRUISE ALTITUDE
A	11 min.	980 + thrust	3920 + thrust	320 MPH	18,000 Ft.
B	8 "	1000	2000	253 "	18,000 "
C	13 "	1000	4000	283 "	18,000 "
D	11 "			400 "	30,000 "
E	11 "			470 "	40,000 "

wise to permit passengers to move about the cabin in very high-speed aircraft. The implication has been that they must of necessity keep the seat belt fastened. Certainly such a procedure would be a considerable deterrent to increased passenger traffic.

Few people recognize the inability of high-speed aircraft to maintain block speeds that approximate the cruising speeds. Recently a study was completed by United Air Lines wherein comparisons between block speeds and cruising speeds were made for a number of contemporary aircraft, two of which were turbojet, one of which was a turboprop, and the other two were relatively new piston-engined transports. Fig. 1 illustrates the results of this study. It will be seen that the two fastest airplanes (turbojets) quickly lose much of their inherent speed for the shorter nonstop distances. In fact, they show little advantage for the first 300 miles. Airplane A is a turboprop, and it can be noticed that it quickly benefits from its increased speed over the reciprocating-engined aircraft. At 1,000 miles its block-to-block speed is 287 m.p.h., as contrasted with its cruising speed of 320 miles.

This study particularly emphasized the problem jet transports have in climbing to high altitudes and then

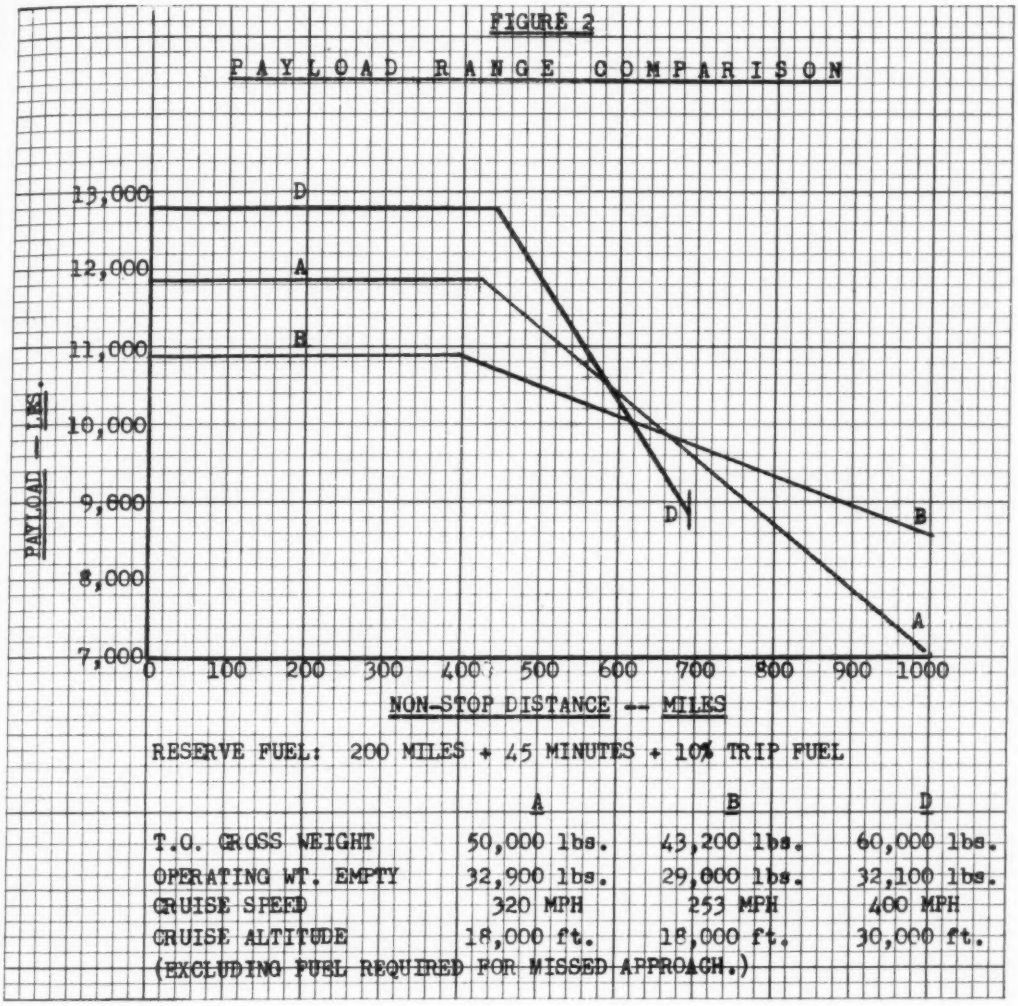


FIGURE 2
PAYLOAD RANGE COMPARISON
PAYLOAD — LBS.
13,000
12,000
11,000
10,000
9,000
8,000
7,000
D
A
B
0 100 200 300 400 500 600 700 800 900 1000
NON-STOP DISTANCE -- MILES
RESERVE FUEL: 200 MILES + 45 MINUTES + 10% TRIP FUEL
A B D
T.O. GROSS WEIGHT 50,000 lbs. 43,200 lbs. 60,000 lbs.
OPERATING WT. EMPTY 32,900 lbs. 29,000 lbs. 32,100 lbs.
CRUISE SPEED 320 MPH 253 MPH 400 MPH
CRUISE ALTITUDE 18,000 ft. 18,000 ft. 30,000 ft.
(EXCLUDING FUEL REQUIRED FOR MISSED APPROACH.)

coming down quickly at the destination. The turboprop suffered somewhat but not to the same degree, and in this case it was possible to make comparisons with the turboprop craft operating at the same altitude as that normally utilized by reciprocating-engined craft.

Fig. 2 represents another phase of this study wherein a pay load-range comparison was made for three of the subject airplanes. It will be noted that the pure jet airplane was not capable of going beyond about 700 miles, when it required a fuel reserve equivalent to 200 miles plus 45 min. plus a 10 per cent trip fuel safety allowance. Airplane A, using turboprops, also lost pay load rapidly for distances beyond 425 miles, because it also used up fuel at a fairly high rate. As would be expected, Airplane B, while carrying the smallest pay load for short distances was, because of the lower fuel consumptions of the reciprocating engine, less adversely affected by offloading of pay load on account of fuel, and therefore it ends up with the greatest amount of pay load for a nonstop distance of 1,000 miles.

Fig. 3 is a comparison of three similar airplanes operating with normal fuel reserves on a 500-mile trip length. It will be seen that the airplanes start out with essentially the same pay load for this length trip. The gross weight of the piston-engined airplane for take-off is 43,200 lbs. However, the take-off gross weight of the turboprop increases to 50,000 lbs., primarily to accommodate the additional fuel required by this type of power plant for the trip. By the same token, the turbojet aircraft must take off with a gross weight of 60,000 lbs.

An inspection of Fig. 3 will now show that, if these same airplanes were to be operated on a 1,000-mile trip length instead of 500 miles, a good portion of the pay load for the turboprop would have to be offloaded in favor of fuel. In the case of the pure jet airplane, it probably could not be operated for a 1,000-mile trip length and still retain the necessary reserves, particularly if it must be assumed that the airplane will sometimes descend from 30,000 ft., make an approach to its original destination, find that it cannot land and then

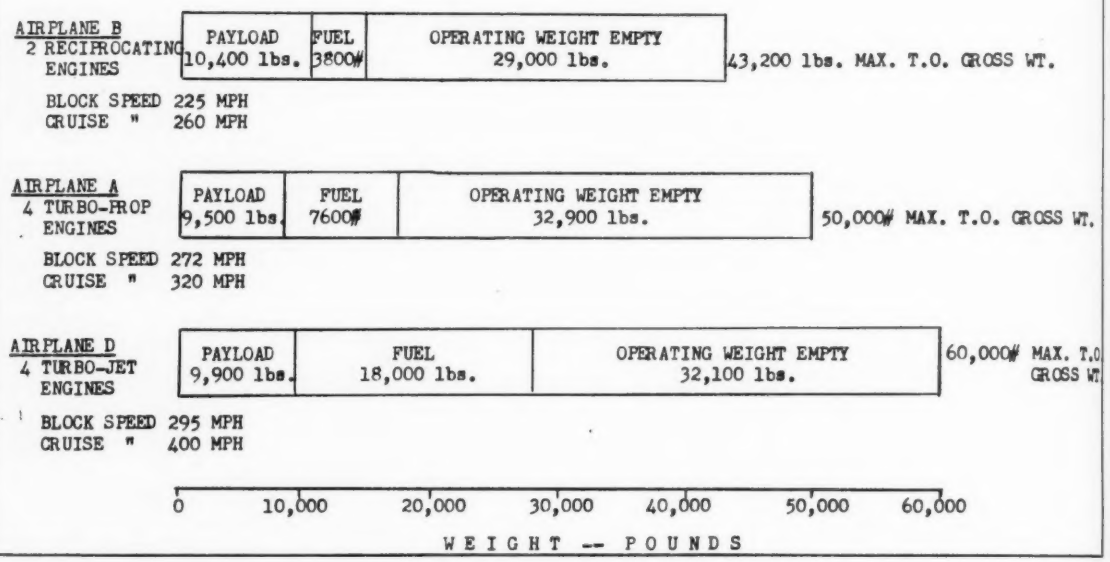
FIGURE 3

WEIGHT BREAKDOWN

THREE 40 PASSENGER AIRPLANES WITH DIFFERENT ENGINE TYPES

500 MILE TRIP LENGTH

(INCLUDING FUEL REQUIRED FOR MISSED APPROACH.)



must climb to the most economical altitude and proceed to its alternate 200 miles away, and still have the legal 45-min. reserve.

This analysis has made it apparent to us that, until the time when specific fuel consumptions for turbines can be improved greatly, the turbojet transport airplane becomes inflexible and its success will depend upon its ability to dispense with reserves and alternates. The turboprop appears to us to be much less critical in these respects and, for that reason, seems to be a more logical next step.

One other point, the variable high-velocity wind conditions that are associated with operation at high altitudes should be emphasized. A study by United Air Lines' meteorologists has resulted in two charts analyzing the average wind velocities that exist at 30,000 ft. in January and in June over certain segments of United Air Lines' route. The high-velocity westerly wind condition for January between Chicago and New York, shown on Fig. 4, is particularly pertinent. It will be noted that westerly head winds can be expected at this altitude and that they will be between 80 and 100 m.p.h. for 19 per cent of the time, between 100 and 120 m.p.h. for 20 per cent of the time, and between 120 and 140 m.p.h. for 18 per cent of the time. Still higher velocities exist for smaller percentages to give a total of 71 per cent above 80 m.p.h. Other route segments suffer from high wind velocities at 30,000 ft. in January but to a lesser extent. The relative frequency of high-velocity westerlies diminishes during the spring to a minimum in summer, but the next chart, Fig. 5, for June, still shows some intense westerlies. It is believed that these conditions essentially demand that any turbine-powered air transports will need to be operated at lower altitudes or make fuel stops under these conditions. Much more favorable wind conditions will often be found for westbound trips during the winter months at low altitudes, as has been indicated in recent reports. (One of these, by Namias and Klapp of the U.S. Weather Bureau, was published in the *Journal of Meteorology*, October, 1949, and carries the title of "Confluence Theory of the High Tropospheric Jet Stream.")

The air transport operators have found the newer postwar four-engined transports—specifically, the DC-6, the Constellation, and the Stratocruiser—to have greatly improved schedule reliability, efficiency, passenger comfort, and independence of weather. The ability of these aircraft to fly long nonstop distances over, around, or under unfavorable weather has been of greatest importance. It is hoped that these recent advantages, achieved as the result of improved flexibility of operation, will not be overlooked by the turbine-engined aircraft manufacturers in their

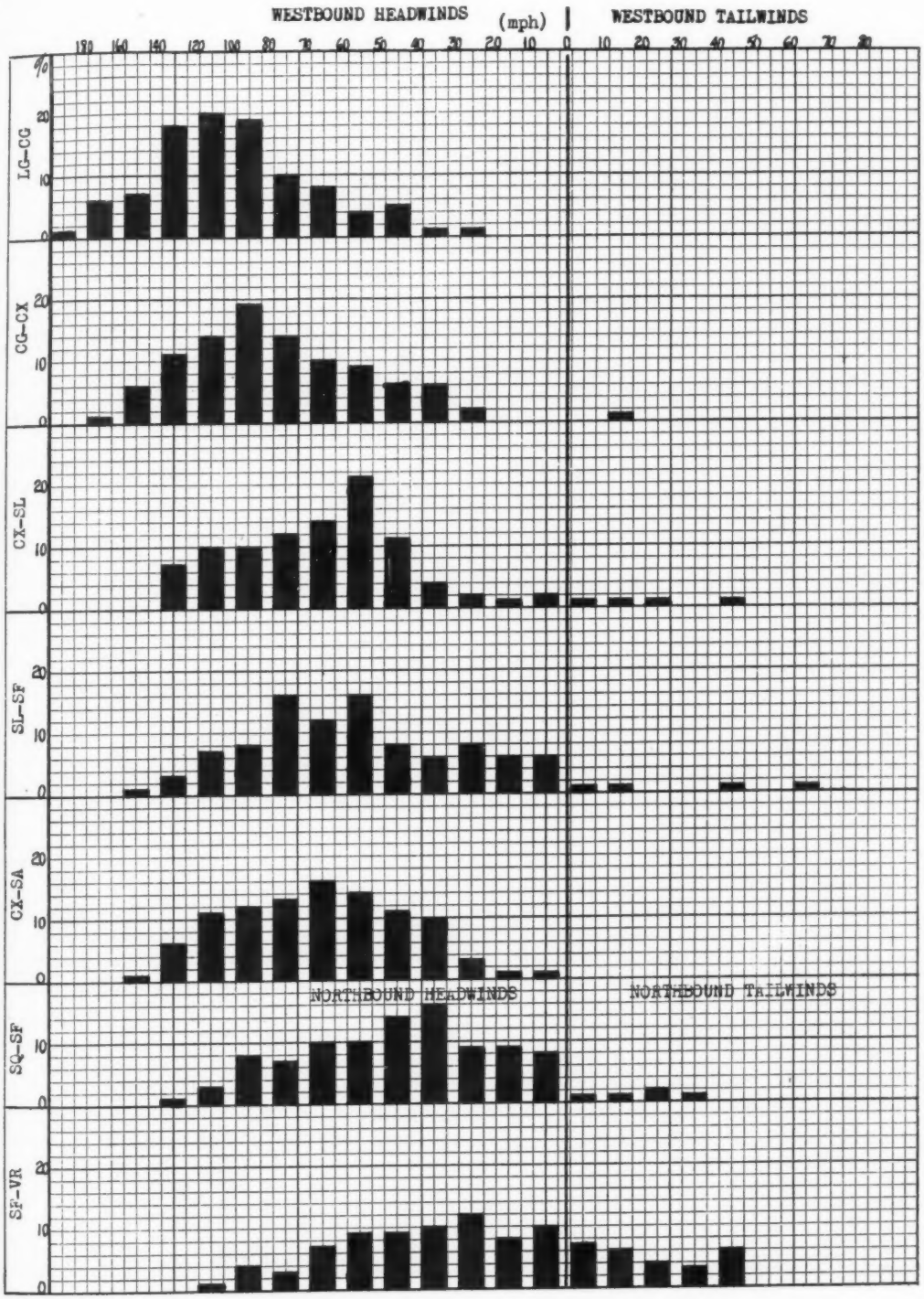


FIG. 4. Sample distribution of wind components at 30,000 ft. during January, 1947.

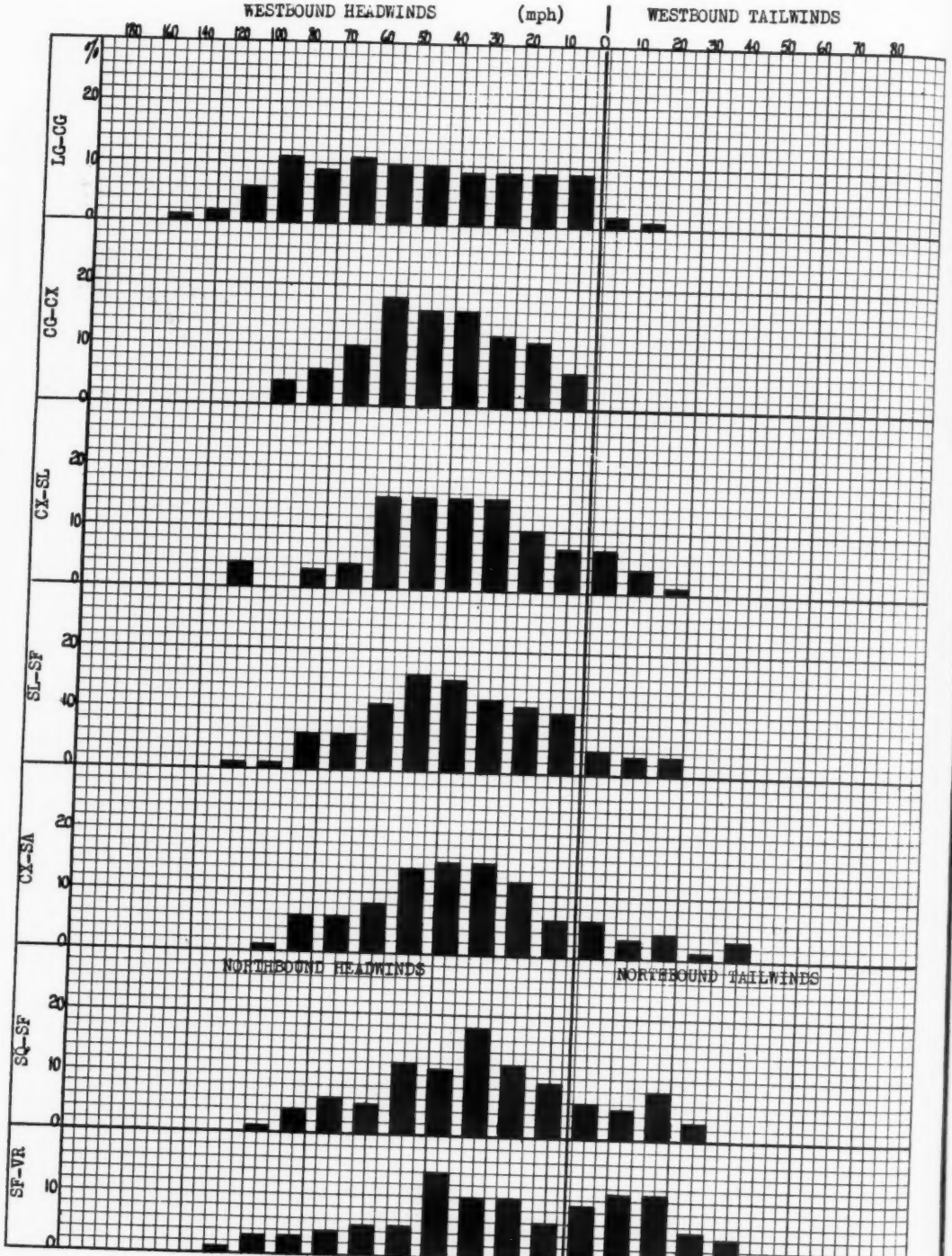


FIG. 5. Sample distribution of wind components at 30,000 ft. during June, 1946.

emphasis on advertised cruising speeds achieved under optimum conditions.

Mr. Russell's lecture gives us assurance that he and his company are approaching the design problems of the turboprop aircraft from a realistic point of view. As a result, we have increasing confidence that the turboprop has an important place in the future of civil air transports and that it offers us a lesser degree of loss of recently acquired "flexibility" than the turbojet.

I consider it a distinct honor to have been asked to discuss this comprehensive lecture given by A. E. Russell. We feel that both he and the Bristol Aeroplane Company, Ltd., are due a vote of thanks for the disclosures they have made concerning design criteria of the turboprop-powered aircraft. We hope that they will be encouraged to continue their work and that, because of this effort, the aircraft industry and the air transport operators will ultimately be able to offer to the public safer, more reliable, and more economical air transportation to every part of the world, having timesaving potentialities and passenger comfort features exceeding the standards of today, to which we point with considerable justifiable pride.

Abe Silverstein (East Coast), Lewis Flight Propulsion Laboratory, N.A.C.A.: The lecturer is to be congratulated for his excellent presentation of a timely and comprehensive treatment of the problems facing the designer of the large airplane and for his exposition of how these problems are resolved. The versatility and scope of knowledge displayed by Mr. Russell in preparing this most interesting and informative paper serve not only to reflect credit on the lecturer but also to illustrate dramatically the many diverse and complex fields of study which the designer must master if he would seek to emulate Mr. Russell's successes.

To the researcher in the aeronautical sciences the paper provides both comfort and despair—comfort in the recognition of the many problems that have been exposed for study and despair in the feeling that these problems accumulate in some multiple ratio of the rate at which they are solved.

The conclusions reached from the analyses and discussions presented in the lecture are believed to be adequately tempered with engineering judgment, and the discusser found no large areas of disagreement.

The author has chosen a most difficult task for the turbine engine and the one that it is not now best qualified to accomplish—namely, that of long range at high altitude. If he had considered it for medium range at medium altitude, his task would have been simplified. Mr. Russell states that airplanes powered by reciprocating engines and by turbine engines could be designed for about equal range at high altitude, with the turbine-powered airplane about 100 m.p.h. faster. If he had considered the modern reciprocating engine, compounded with exhaust turbines feeding power into the crankshaft, it is believed that his analysis would have revealed a different answer, for it can be shown that, at altitudes of 35,000 and 40,000 ft., the range of an airplane equipped with a compound engine is greater than that of the airplane powered with turbine engines that operate under the modest values of pressure ratio, peak temperature and efficiencies assumed in Mr. Russell's paper. Further, the cruising speeds of the airplanes equipped with compound and turbine engines would not be greatly different.

Despite the author's failure to make a comparison of the present turbine engine with the compound engine, which would have revealed the inferior position of present-day turbine engines with respect to range at high altitude, the conclusion he has drawn that large prototype turbine-powered airplanes should be constructed is believed to be a sound one. The turbine engine must be accepted now as a long-range engine on the basis of its future potentialities, which appear brighter than those of the compounded reciprocating engine. The gains to be made in the next few years from increasing the turbine cycle pressure, peak cycle temperature, and component efficiencies will place the turbine in a sound competitive position on the basis of performance at altitude with the attendant advantages of reduced vibration, larger power units, and simplification—swinging the pendulum in favor of the turbine engine.

Since structural considerations, safety, and comfort in rough air plague the engineer as speeds increase, any reduction in the experience of rough air is worth-while. Perhaps the value of radar for this purpose has been underestimated. The use of radar has been shown to reduce gust experience through the avoidance of some turbulent clouds, and on the average, therefore, the use of radar would be expected to be beneficial. As information is collected, it is not inconceivable that there may be weight allowances for the use of such devices where convective clouds are a major operational problem. While clear air turbulence is a factor, the more severe gusts appear to be associated with convective cloud formations.

The introduction of turbine-propeller power plants with higher operating altitudes may result in reduced gust load experience, with consequent benefits as experience is gained. It would be expected that, by virtue of avoidance of clouds that cause difficulty at the lower levels, the intensity of gusts experienced at cruising speed might be appreciably lower for the higher level operations. The current ICAO requirements recognize this possibility. Future research and data collection for operations visualized by the author may result in radical changes in the statistics of gust load experience.

Gust alleviation still poses many problems of flying qualities, flutter, structural loads, and comfort, as so well recognized by Mr. Russell. With regard to ultimate strength, maneuver load factors impose a minimum strength that may not permit much gain to be

realized from gust alleviation. From the point of view of comfort, control, and fatigue life, it offers the greatest benefits, since it reduces all gust effects. If fatigue is a serious factor in the wing design, alleviation can be the difference between a good and a bad airplane.

In passing, some problems of alleviators not mentioned by Mr. Russell are the fact that a gust detector on the fuselage detects the local value of the gust and that, when the myriad gust shapes are considered, the possibility of a spurious signal due to a gust at the detector opposite to that acting on the wings is a matter of concern. The variation of alleviation and the time lag between detection and action with speed are other questions that will need careful consideration. It is believed from Mr. Russell's remarks that these items are under consideration, together with many others.

The important effect of air damping on elastic response in gusts discussed by Mr. Russell has been recognized. It is felt, however, that generalization at this time is not warranted, since, as speeds are increased to sufficiently high values, the air damping may reach a maximum and decrease drastically at any higher speeds.

Although not mentioned, the added conditions to be investigated by the designer are probably of serious concern as to engineering costs. The hope of simple solutions through analogs or models bears directly on this problem. The determination of the proper aerodynamic and structural coefficients remains a serious problem. As aircraft get larger and carry more mass in the wing, the precision of determining dynamic airplane response must become greater.

It is of interest to note that in the discussion of flutter and vibration the lecture brings to bear on the problem several combined modes of attack, including flight testing, normal mode calculations, use of models, analogs, and high-speed digital computing machines.

The discussion on gust detection and alleviation and the airplane structural problems is based on information obtained from Philip Donely, of the N.A.C.A. Langley Aeronautical Laboratory.

Let me close by assuring Mr. Russell that all of us have learned a great deal from his work and its discussion, and I offer him my best wishes for continued success in his pioneering work on large turbine-powered airplanes.

Frank W. Fink (West Coast), Chief Engineer, Consolidated Vultee Aircraft Corporation: It is a pleasure to be asked to comment on Mr. Russell's very fine paper. Just 2 months ago, I had the pleasure of visiting the Bristol Aeroplane Company in England where Mr. Russell showed me around.

The subject of Mr. Russell's paper is of extreme interest at the present time in the United States, and I should like to congratulate him on his splendid presentation in this field. Mr. Russell states that his paper embraces a smaller field than that of the scientists who have made papers for earlier lectures in this series; however, I believe he has covered rather a broad field. The paper touches on many points, any one of which could be the subject of an entire presentation—such as turbine engine material, turbine blade cooling, turbine using heat-exchanger cycle, optimum wing loading, power loading, span loading for a turboprop airplane, etc.

One point, however, that he did not bring up but which is of great interest in this country is the control of the turboprop engine. I mention it here merely because the control for the type of turboprop Mr. Russell describes in this paper can be extremely simple. Since the turbine stage that drives the propeller is in no way mechanically interconnected with the turbine stages that drive the compressor, there are no problems of high torsional shaft stresses caused by an unbalance between the prop load and the compressor load.

Mr. Russell confines his paper to the long-range turboprop transport airplane. I believe something should be said for use of turboprop in a short-range transport airplane.

It is true that the fuel consumption will be somewhat higher than a reciprocating engine for some time to come, but it will be much lower than for a jet engine of the size that will give equivalent take-off thrust and when operated at medium altitudes.

It is pointed out in the paper that for endurance below 5 hours, a comparative low compression ratio engine may be used. A lower compression ratio turboprop appears also to fit in better with the requirements of a short-range transport, in that it will operate efficiently at a somewhat lower altitude, say 20,000 to 25,000 ft., and is less susceptible to high ambient temperatures. A short-haul airplane is likely to have to operate continuously during some seasons of the year under tropical or hot-day take-off conditions.

It is believed that a turboprop engine can be operated just as economically on short-haul airplanes as a reciprocating engine, since the engines have much lower maintenance costs and should eventually be just as economical to build and, further, since the extremely low vibration should greatly decrease airplane maintenance costs—to say nothing of increased passenger travel generated by the increased comfort. A turboprop also has a distinct advantage over turbojet short-haul transports, in that short-haul transports must, by necessity, operate from many airports that will not see improvements necessary for jet transports for many, many years to come. Propeller-driven airplanes will be able to take off on shorter fields, land on shorter fields because of reversing, and cope better with traffic conditions at fields that are not completely equipped for blind landings.

One interesting point that Mr. Russell made was that of the present range of propeller-driven sizes now being manufactured or developed; the specific weight of similar designs appears to be roughly constant and,

therefore, in certain cases it is beneficial to couple two small engines together in order to make a more compact and more suitable installation for burying in a wing. This happy condition could also be used to great advantage in short-haul transport airplanes, since an airplane could be designed with four engines paired together so that only two nacelles and two propellers would be required. This would give the compactness and aerodynamic efficiencies of a two-engined airplane with the reliability and one-engine-out performance of a four-engined airplane.

E. G. Stout (West Coast), Assistant to the Chief Engineer, Consolidated Vultee Aircraft Corporation: Mr. Russell pointed out that there is no advantage in cutting out turbine units on multiunit airplanes. His conclusions are based, however, on the ideal condition of cruising at optimum altitude, where the full power of all units can be used efficiently from a fuel economy standpoint. For the condition where a specified altitude must be held, as in the case of radar search with a military airplane, sizable improvements in range are obtainable by appropriate selection of units. For example, the Convair XP5Y-1 T-40 propeller-turbine-powered long-range flying boat, which for certain radar search missions is operated continuously at 10,000 ft., shows increases in range as follows:

All unit operation	Basis
75 per cent units operation	30 per cent increase
50 per cent units operation	65 per cent increase
50 to 25 per cent units operation	105 per cent increase (or over double the range obtainable without cutting units)

Of course, the economical cruising speed drops as the range is increased (following the usual speed-range relation).

E. C. Wells (West Coast), Vice-President—Engineering, Seattle Division, Boeing Airplane Company: Mr. Russell should indeed be commended for his courage in selecting the broad field of aeroelastic phenomena and dynamic loads for a portion of his paper. Considering the scope of his selected subject, he has come forth with a thorough and straightforward discussion of the problems and the general significance of design trends.

As he mentioned, the dynamic and aeroelastic aspects of the structural loads problems have only recently become of importance in the design of basic structure. Thus, as in any new and rapidly expanding phase of design, we find ourselves proceeding into somewhat unfamiliar territory with little more than a reasonable understanding of the physical relationships involved together with analytical and experimental tools through which we can increase this understanding.

Today we find ourselves introducing the additional parameter, time, into our structural loads concepts. Unfortunately, a large percentage of the experimental data that formed the foundation for the pseudostatic structural design criteria was taken without adequate recognition of the future importance of time as a variable in the loads problem. However, this foundation material should neither be arbitrarily discarded nor be blindly accepted as applicable. Rather, it should be re-evaluated, where possible, in the light of the new problem. Since time is now of significance, we are interested not only in the probability of occurrence of a load but also in the probability that it will occur with a given time history. As Mr. Russell has shown, the time history of the load application may be of equal or greater importance than the close definition of the maximum magnitude. Study of available gust data will reveal investigations of several airplanes from which graphs similar to Fig. 1 may be obtained. It will be noted on this plot of true gust velocity vs. gradient distance, that high velocity gusts are not confined to a narrow band of gradient length. X gusts of approximately Y velocity occur. There is therefore a probability of X divided by the total number that this velocity will be attained. However, the probability of any given gradient length at this velocity is considerably reduced. This difference of probability should logically be reflected in the design.

Another important feature of the gust problem which warrants some consideration is the velocity variation along the span and fore and aft during the passage of the airplane. The generally accepted linear chordwise gradient with the maximum velocity sustained and a constant velocity spanwise at any instant is open to question. The importance of the chordwise variation of velocity and therefore the time history of the externally applied loads can be seen on Fig. 2. Here is shown the dynamic response factor for the linear gradient with sustained maximum and for comparison the response to a force that increases linearly to a given value and thereafter decreases to zero at the same rate. The curves represent the response of a simple single degree of freedom lumped system without damping. The differences are pronounced for the shorter gradients, particularly those for which the sustained type loading indicates the greatest dynamic effect. Considerable evidence exists to support the use of forcing functions of approximately the triangular time history rather than the sustained type for the usual gust. Again, probabilities become important.

It is interesting to carry this triangular velocity distribution concept one step further to examine its relationship to the spanwise load distribution. As Mr. Russell has pointed out, statistically an airplane will enter a gust asymmetrically much more often than not. If the velocity variations in the atmosphere are indeed somewhat triangular, this would certainly be expected. The few measurements that have been attempted indicate a triangular or sinusoidal type variation to be equally as applicable to the spanwise

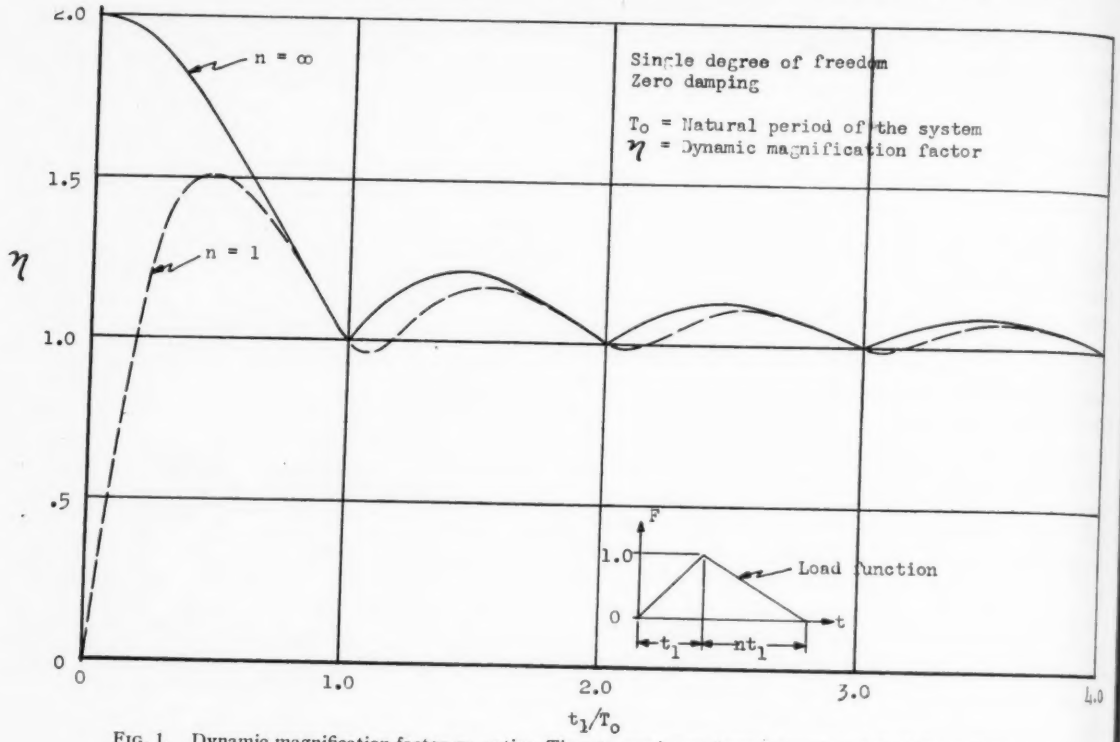


FIG. 1. Dynamic magnification factor vs. ratio: Time to maximum force/natural period of the system.

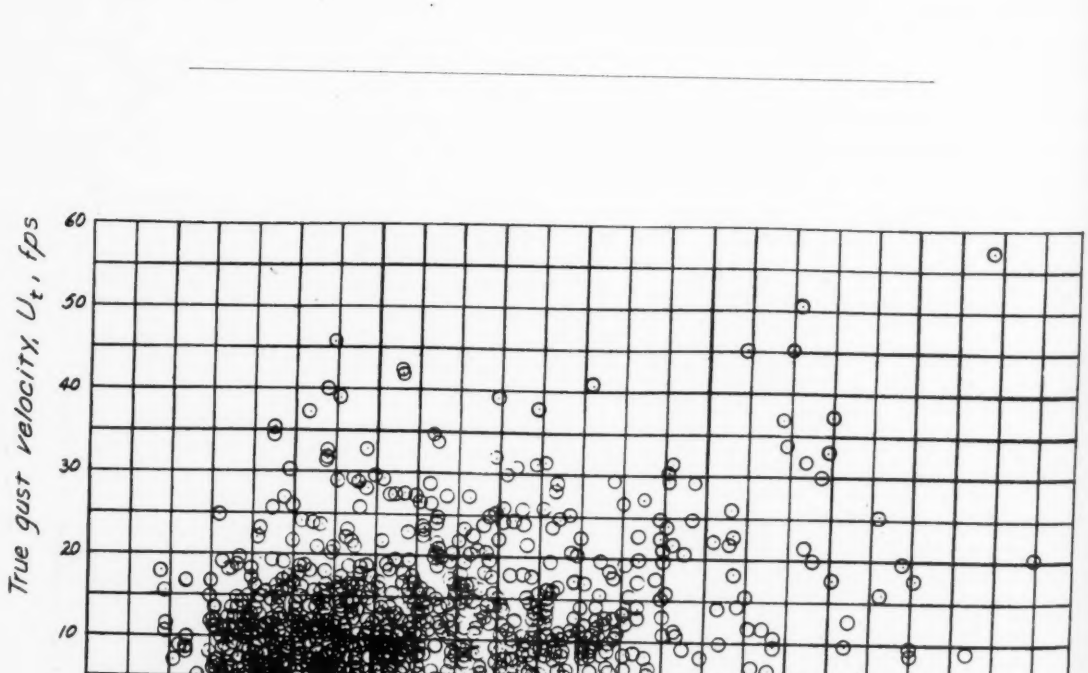


FIG. 2. True gust velocity U_t as a function of the gust-gradient distance, H. (Data obtained in 1942.)

velocity variation as to that in the direction of flight. It would, in fact, be somewhat naïve to assume a difference between the distributions in the two perpendicular directions, since it is unlikely that gusts would have any particular geometric alignment with the direction of flight. The distribution will have important influences on both the pseudostatic loading and the response of the elastic airplane.

As with any problem, the best solution is the elimination of the problem. Certainly this is true of gust loads. Thus, the development of gust alleviators is a logical step. The development of aircraft for flight in the relatively high subsonic speed range has resulted in the use of sweptback lifting surfaces to obtain their favorable effects on compressibility phenomena. Fortunately, sweepback also produces a significant reduction in gust loads.

The major alleviating effects of swept surfaces are twofold. As is well known, the slope of the lift curve is lower than that of the equivalent straight surface. Hence, the effective angle of attack change due to a given gust will produce less load change or acceleration on a sweptwing airplane. Second, bending of the wing produces alleviating effects similar to those of a skew hinge. The trend can be conveniently illustrated by considering conditions of stabilized flight. Fig. 3 shows typical lift curves for a wind-tunnel model with the wings deflected to the several positions corresponding to flight at different load factors and rigidly restrained in these positions as the angle of attack is varied. The curves therefore individually represent lift vs. angle of attack for a rigid wing with a particular warpage of the plane of zero lift lines, and their slopes would include the loss due to sweepback mentioned previously. On each curve, a circle has been drawn at the approximate lift coefficient appropriate to the load factor, assuming a given constant air speed. The lift change would tend to progress along the dashed curve between these points in a gust. Hence, the elastic deflections produce an additional effective reduction in the slope of the lift curve.

Examining these elastic effects further, it is seen that the major changes in lift occur near the tips of the wing for the usual wing deflection shape—a favorable arrangement structurally. Not only do the elastic deflections of a sweptwing reduce the gust loading, but they also serve to shift the spanwise center of pressure of the remaining load inward.

In some respects, sweepback is unfavorable as a gust alleviator. For example, for typical gusts encountered symmetrically, the forward areas of the wing in the vicinity of the root enter the gust first with a resulting tendency to pitch the airplane in a direction to increase the effectiveness of the gust. Similarly, the inward shift of the aerodynamic load due to elastic deflection is accompanied by a forward shift. This latter phenomenon is closely related to the short-period stability of the airplane, however, and therefore

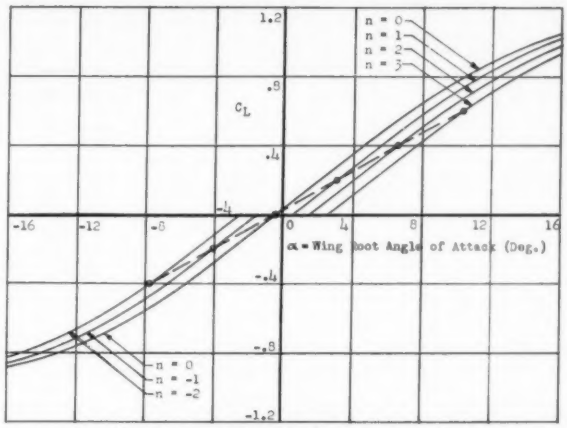


FIG. 3. Effect of wing deflections upon dC_L/d.

is somewhat compensated for in the selection of permissible center-of-gravity positions. Thus, it is found that the usual swept surface is also conveniently a powerful gust alleviator without the ever-present disadvantages of a mechanical system.

It is felt that Mr. Russell's rather unfavorable views on the value of testing scaled flutter models are not entirely justified. At any rate, our own experience in that field has led us to believe that an extensive model testing program is essential as a supplement to analytical flutter studies. Of course, it must be admitted that full-scale Reynolds Numbers cannot be approached with models of reasonable size. However, the limited experimental data available in this country tend to suggest that Reynolds Number is not of primary importance in flutter phenomena, except possibly at transonic speeds where separation and compressibility effects may be fundamentally interrelated. Certainly, the existing theory of oscillating airfoils is not sufficiently general to encompass scale effects, and it may be argued that model data are no more suspect than analytical results in that respect. Also the critical flutter modes of a large, flexible airplane generally involve simple distortions of the primary components; for that reason it is felt that a simplified model structure may be employed without consideration of the effects of shear deflection. Actually the number of normal modes of the complete airplane which involve only simple deformations of the primary components is relatively large; hence, it is possible with a simplified model to obtain a more comprehensive representation of the dynamic characteristics of the airplane structure than would be possible at the present time in an analytical investigation of reasonable scope. Our usual practice is to conduct both analytical and experimental investigations concurrently during the development of a new model. Fig. 4 illustrates some of the essential features of one of our flutter models. It will be noted that the primary wing structure consists of a single metal spar of solid section, suitably proportioned to give appropriate distributions of

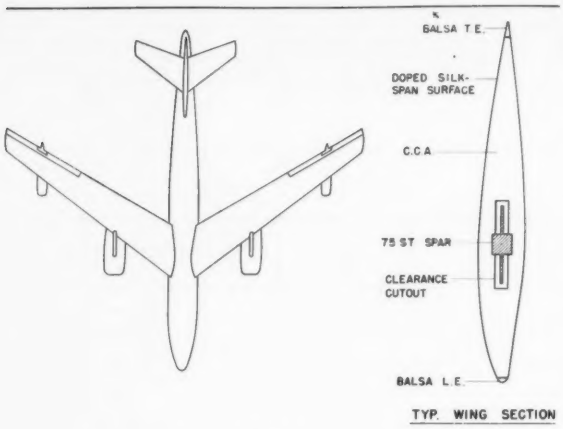


FIG. 4.

flexural and torsional stiffnesses. A similar arrangement can be adopted for the fuselage. The contour is accomplished by the use of cellular cellulose acetate of low density or balsa wood so arranged as to provide no additional stiffness over the basic spar.

Unquestionably, one of the greatest advantages to be derived from a flutter-model test program is the ability to determine relatively quickly the consequence of design modifications and changes in flight conditions or airplane loading. Because of the simplified construction, it is not difficult to manufacture new components incorporating changes of stiffness or of geometric form. Springs of different stiffnesses are usually provided in order to investigate the effect of variations in control surface natural frequencies, and variations in airplane altitude are simulated by adding or removing weight from the model. With present computational facilities we have found that the parallel analytical program of flutter studies simply cannot keep pace with the experimental program. In spite of its limitations the flutter model does provide answers in time to be of use during the design of an airplane, thus materially reducing the likelihood of costly changes during the later stages of development.

Also, it has been our experience that flutter model tests generally reveal the existence of critical modes that might be overlooked otherwise because of the understandable tendency to oversimplify the analytical representation of the airplane. These tests have served to reinforce substantially our conviction that a large, flexible airplane structure should be regarded as a single, interconnected dynamical system. In one particular instance, a moderate change in elastic restraint of the elevator served to introduce a very large wing vibration component into a flutter mode, in addition to the tail and fuselage motions that were already present. As a matter of fact, four distinct flutter modes were found within the practical range of elevator frequencies in this case. The experimentally determined flutter speed is shown as a function of elevator frequency in Fig. 5. Experiences of this kind serve to lessen further our confidence in the validity of any sort of simplified flutter criteria.

In the design of our recent models, we have also maintained dimensional similarity of elastic deformations due to weight and steady air loads in order to investigate other aeroelastic phenomena as well as flutter. Recently, consideration has been given to the possibility of testing aeroelastic models at high subsonic Mach Numbers in order to determine compressibility effects. Unfortunately, it appears to be impractical to maintain dimensional similarity of flutter and the other aeroelastic characteristics, as well as compressibility effects, with a model of reasonable size.

I certainly do not mean to imply that the analytical investigations should be abandoned or reduced in scope. On the contrary, we are in agreement with Mr. Russell with regard to the desirability of exploiting the modern developments in electronic computers to the fullest extent for the analysis of aeroelastic phenomena. It is anticipated that these machines will ultimately permit us to conduct a sufficiently complete analysis of the entire airplane as a single, interconnected, dynamical system. Our own studies in this field tend to favor the digital computer. In order to employ an analog computer for flutter analysis, it is necessary to accomplish a considerable amount of preliminary computational work before the problem can be put on the machine. Also, the problem of introducing such refinements as compressibility and aspect ratio corrections into the analog computer is difficult and possibly impractical. These modifications, as well as any others that may be considered necessary in the light of future research, could be handled without particular difficulty on a digital machine.

Mr. Russell's comments on ground vibration testing of large aircraft have suggested an ingenious solution to a difficult and important problem—namely, the provision of a supporting system that will permit

(*Continued on page 122*)

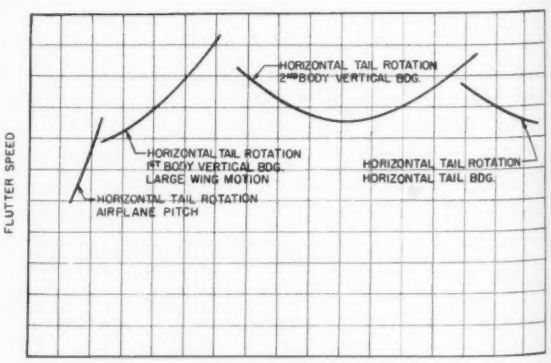


FIG. 5. Flutter speed vs. elevator frequency.

Application of Simultaneous Equation Machines to Aircraft Structure and Flutter Problems

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SUMMARY

Flutter, divergence, and control surface reversal are generally known as the aeroelastic problem. Not only aerodynamic but also elastic and inertia forces are involved. Considerable mathematical effort is required along with the development of a proper physical conception to deal with this problem. It is used as an example in this paper, since the time expended on this problem in modern aircraft design led the authors to the study of the linear algebraic simultaneous equation machine. It also represents a typical aircraft engineering problem that may be solved by such a machine.

TYPES OF PROBLEMS

ENGINEERING problems may be classified as determinate or indeterminate. The simultaneous equation machine bears the same relationship to algebraic indeterminate problems as the slide rule to determinate ones. Indeterminate problems may be further subdivided into simple indeterminate, in which only one condition of equilibrium exists, and stability problems, where a multiplicity of solutions exists.

SIMPLE INDETERMINATE PROBLEMS

These may be expressed by ordinary simultaneous equations of the form:

$$A_{11}X_1 + A_{12}X_2 + A_{13}X_3 \ldots = B_1$$
$$A_{21}X_1 + A_{22}X_2 + A_{23}X_3 \ldots = B_2$$
$$\ldots\ldots\ldots$$

Some of the numerous aircraft engineering problems that may be so expressed are: (1) statically indeterminate structural problems; (2) forced vibration problems; (3) span load distribution; and (4) control surface effectiveness.

Some of these problems illustrate an additional aspect of equation machines, in that integrations may be performed by breaking them down into increments. A single integration appears explicitly in the control surface effectiveness problem, and they are implicitly contained in some of the others.

The equation computing machine solves these problems by generating, multiplying, and adding electrical voltages in exactly the same form as the equations.

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STABILITY PROBLEMS

The second class of problems is much more difficult than the first because of the multiple conditions of equilibrium possible, and they may be expressed mathematically in the form:

$$(A_{11} - \lambda)X_1 + A_{12}X_2 + A_{13}X_3 \ldots = 0$$
$$A_{12}X_1 + (A_{22} - \lambda)X_2 + A_{23}X_3 \ldots = 0$$
$$\ldots\ldots\ldots$$
$$\ldots\ldots\ldots$$

Here the additional variable λ introduces n possible solutions to the equations, representing n possible equilibrium conditions, where n equations are involved.

Aircraft engineering problems in this class are, for example: (1) buckling; (2) free vibration problems; (3) divergence; (4) flutter; and (5) aero stability, static and dynamic.

In order to handle these problems, an additional set of multipliers is supplied which are ganged together and vary λ as required by the equation.

This λ term is also useful in some of the first class of indeterminate problems, although it is not essential to their solution as in the second class.

MISCELLANEOUS PROBLEMS

Other problems of a more general nature may also be solved, such as:

(1) Algebraic polynomial equations (a machine with n equations will solve an nth order equation. The λ term is required.)

(2) Multiplication and inversion of matrices. (These require the solution of n ordinary simultaneous equations, in general. A complete matrix multiplying and inverting machine, that could perform this in one instead of n operations could be built on the same principles.)

(3) Step-by-step solution of variable coefficient and certain nonlinear problems.

COMPLEX NUMBERS

Many of the above problems require complex numbers for their representation. The machine used as a basis for this paper is based on real numbers and handles complex equations by expanding them to twice the number of real equations. Thus,

$$(a + ib)(x + iy)^2c + id$$

is equivalent to

$$ax - by = c$$
$$bx + ay = d$$

The Aeroelastic Problem

The aeroelastic problem consists of the combined aerodynamic, elastic, and inertia properties of an aerodynamic surface which manifest themselves in the phenomena of torsional divergence, control surface effectiveness and reversal, and flutter. A corollary to these are the vibration modes and frequencies of the structure. The difficulties in dealing with these problems may be seen from the fact that we must not only represent mathematically a complicated set of structural and aerodynamic forces but also their performance as a time history because of the oscillatory nature of vibration phenomena.

An aircraft structure has continuously distributed properties. For analysis purposes these are sectionalized and represented as lumped parameters strung together to approximate the true structure. The aerodynamic forces are dealt with in a similar fashion. Variation with time is dealt with by assuming uniform sinusoidal motion, and the forces may then be represented by rotating vectors with given magnitudes and phase relationships. If it is further assumed that all of the forces involved are linear with their related displacements, the problem is then formulated so that the solution may be obtained. This composition of the problem is not ultimate, and improvements in it are being made; however, it is the present working theoretical basis.

The above assumptions have made the problem solvable and even simple in the physical sense; however, they have not left it easy to obtain answers. The expression of the above characteristics as force equations leads to simultaneous equations of orders varying with the complexity of the problem from two complex equations in simple problems up to eight or more in cases requiring the additional refinement. The solution of these simultaneous equations, while in principle very simple, are, in fact, laborious, and some of the best mathematicians and engineers have applied their efforts to finding short cuts to reduce the labor involved. One approach is the use of an analogy computing machine, set up to represent the equation coefficients.

It was believed that such machine solutions represented a powerful and worth-while approach to the problem of saving labor and giving a physical feeling for the relative importance of various parameters in specific cases—that is to say, their variations are easily explored. To this end a simultaneous equation machine was constructed. A four-element machine was chosen because it was the smallest machine that would show up the essential characteristics of the problem. The problems solved on it are thus of a simple nature, but they fully illustrate the principles involved in the solution of the more refined cases and thus form an interesting evaluation of the capabilities of this method.

The Simultaneous Equation Machine

The equation machine solves the problem by multiplying and adding voltages in exactly the same form as the mathematical expressions. A potentiometer circuit is used for multiplication, $R_2 >> R_1$ and a parallel circuit for addition, $R >> Z_g$.

The machine thus multiplies directly and divides by the inverse process by adjusting the multiplier (unknowns) until the solution is obtained.

Basic Classifications

There are four basic types of simultaneous equation analogy machines. This classification does not depend on whether or not the machine is electrical, mechanical, hydraulic, or based on probability multiplication. Any physical process with a multiplicative law that is also capable of addition may be used as a basis for such a machine. The classifications are: (1) reversible, manual; (2) reversible, automatic; (3) semireversible, manual; (4) semireversible, automatic.

Whether a machine is manual or automatic is a matter of design refinement rather than the mathematical principles on which it is based. The reversibility or the degree of reversibility is, however, a part of the mathematical design of the machine.

The title "reversible" is used because machines based on reversible physical processes—such as the lever, gear, or transformer—immediately indicate the complete solution. A synthetic mathematical reversibility may be supplied to irreversible processes to achieve the same results. This may be either automatic or manual. If automatic, the answer is immediately indicated; if manual, as is this machine, convergence to it is guaranteed. In this particular calculator, this is accomplished by minimizing the sum of the absolute values of the errors of each equation. This can only be zero when all equations are satisfied. The absolute value is taken by a rectifier before adding the error signals.

Semireversible machines are based in an approximation to the reversible process. The most common of these approximations is mathematically equivalent to the Gauss Sidel iteration procedure. These machines will converge for some problems and diverge for others, whether manual or automatic. These machines may be given the "synthetic" reversibility by considerable mathematical preparation of the problem. This also may be done in the machine at the expense of halving its capacity.[1]

Method of Solution

A problem is prepared for the machine by placing it in simultaneous equation form. The equations are then

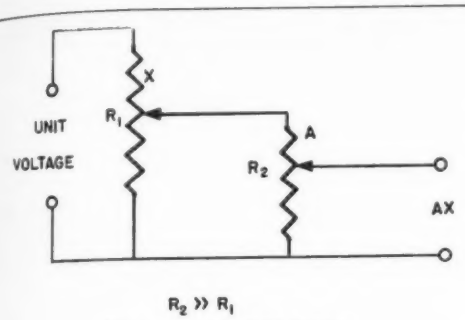


FIG. 1. Multiplying circuit.

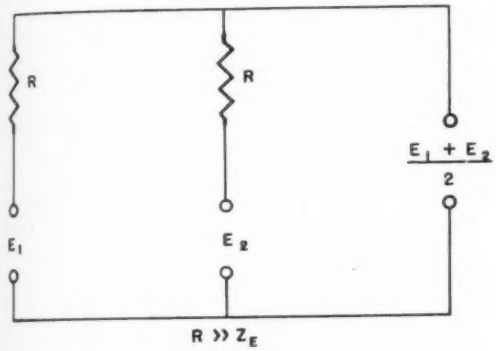


FIG. 2. Adding circuit.

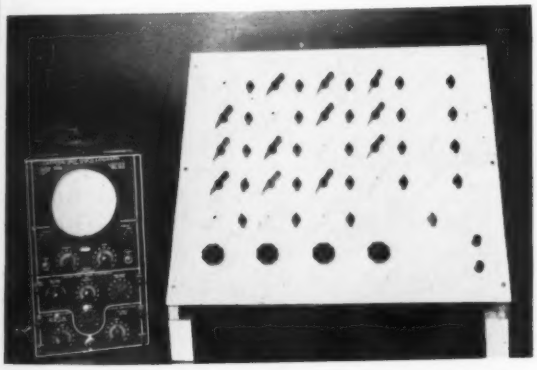


FIG. 3. Experimental simultaneous equation calculating machine.

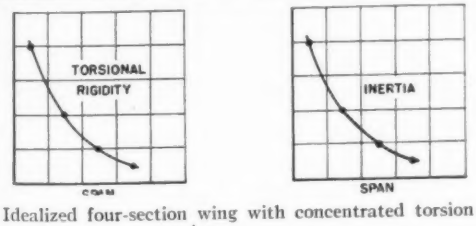


FIG. 4. Idealized four-section wing with concentrated torsion springs.

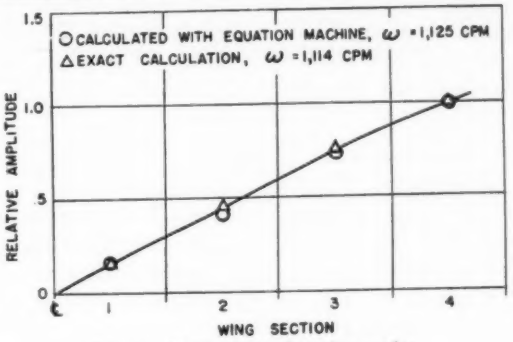


FIG. 5. Comparison of torsion modes.

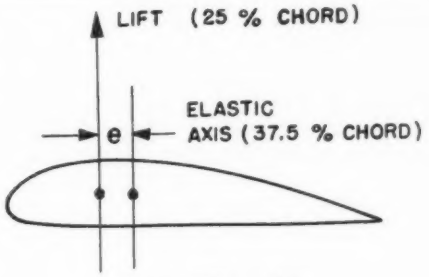


FIG. 6. Wing section.

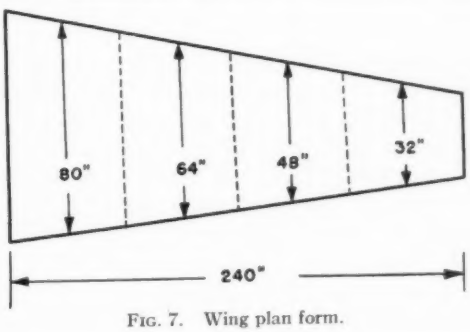


FIG. 7. Wing plan form.

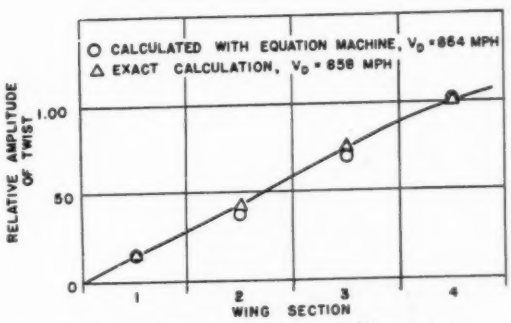


FIG. 8. Comparison of torsional divergence.

multiplied by such factors as necessary to bring the largest coefficients near 1, since the machine is calibrated to handle numbers between +1 and −1. The coefficient dials are then set, and the unknowns are manipulated until the output signal is zero.

In frequency problems, the ratio of the unknowns (amplitudes) is the desired information, while the overall value of the unknowns (amplitudes) is arbitrary, so one of them is fixed and the diagonal term is manipulated in its place. In the flutter problem, the phase and amplitude of one component are fixed, and the diagonal and one of the constant terms are used in their place. The sign and magnitude of this constant term are indications of the stability of the system at the conditions of the computed point. There are many possible variations to the above technique.

In general, because of the full reversibility of this machine, any n of the dials—chosen from the unknowns, coefficients, constants, or λ term—may be manipulated as unknowns, depending on the information desired. Such freedom of procedure has been little discussed mathematically and is believed to have great possibilities.

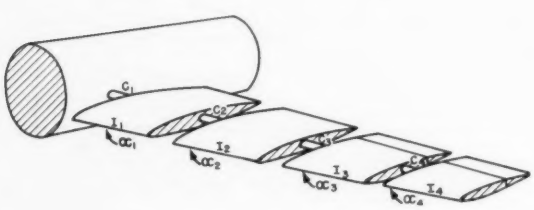
Illustrative Problems

These problems, solved by the four-equation machine, are illustrative in character. For the sake of concreteness, an actual wing is invisaged, and typical properties are assigned to it, as shown in Figs. 4, 6, 7, and 9.

For many practical cases, it would be desirable, of course, to divide the wing into more than four sections and to otherwise refine the treatment used in these sample problems. It is believed, however, that the basic principles of the use of the machine are adequately covered and that further detail refinements in technique will be sufficiently obvious of those familiar with these problems.

The following problems are solved, and the solutions are compared to those obtained by a hand calculating machine: (1) first natural torsion vibration modes and frequency; (2) torsional divergence; (3) aileron effectiveness; and (4) wing bending—torsion flutter.

Wing Properties Used in Illustrative Torsion Problems



Idealized Four-Section Wing with Concentrated Torsion Springs

Spring Constants	Mass Inertias
$C_1 = 20 \times 10^6$	$I_1 = \dfrac{4 \times 10^6}{28.41}$
$C_2 = 10 \times 10^6$	$I_2 = \dfrac{2 \times 10^6}{28.41}$
$C_3 = 5 \times 10^6$	$I_3 = \dfrac{1 \times 10^6}{28.41}$
$C_4 = 2.5 \times 10^6$	$I_4 = \dfrac{0.5 \times 10^6}{28.41}$

Deflections

The twists along the wing for general applied torques may be written:

$$\begin{aligned}\alpha_1 &= (1/C_1)(T_1 + T_2 + T_3 + T_4)\\ \alpha_2 &= \alpha_1 + (1/C_2)(T_1 + T_2 + T_3)\\ \alpha_3 &= \alpha_2 + (1/C_3)(T_3 + T_4)\\ \alpha_4 &= \alpha_3 + (1/C_4)(T_4)\end{aligned}$$

or

$$\left.\begin{aligned}\alpha_1 &= 0.05T_1 + 0.05T_2 + 0.05T_3 + 0.05T_4\\ \alpha_2 &= 0.05T_1 + 0.15T_2 + 0.15T_3 + 0.15T_4\\ \alpha_3 &= 0.05T_1 + 0.15T_2 + 0.35T_3 + 0.35T_4\\ \alpha_4 &= 0.05T_1 + 0.15T_2 + 0.35T_3 + 0.75T_4\end{aligned}\right\} \times 10^{-1}$$

(a) *The Frequency Problem*

In this case, the torques are given by the inertia forces:

$$\left.\begin{aligned}T_1 &= W^2 I_1 \alpha_1\\ T_2 &= W^2 I_2 \alpha_2\\ T_3 &= W^2 I_3 \alpha_3\\ T_4 &= W^2 I_4 \alpha_4\end{aligned}\right\} \times 10^6$$

Substituting, we have

$$\begin{aligned}1/W^2\alpha_1 &= 0.20\alpha_1 + 0.10\alpha_2 + 0.05\alpha_3 + 0.025\alpha_4\\ 1/W^2\alpha_2 &= 0.20\alpha_1 + 0.30\alpha_2 + 0.15\alpha_3 + 0.075\alpha_4\\ 1/W^2\alpha_3 &= 0.20\alpha_1 + 0.30\alpha_2 + 0.35\alpha_3 + 0.175\alpha_4\\ 1/W^2\alpha_4 &= 0.20\alpha_1 + 0.30\alpha_2 + 0.35\alpha_3 + 0.375\alpha_4\end{aligned}$$

or calling $1/W^2 = X$,

$$\begin{aligned}(0.30 - X)\alpha_1 + 0.10\alpha_2 + 0.05\alpha_3 + 0.025_4 &= 0\\ 0.20\alpha_1 + (0.30 - X)\alpha_2 + 0.15\alpha_3 + 0.075\alpha_4 &= 0\\ 0.20\alpha_1 + 0.30\alpha_2 + (0.35 - X)\alpha_3 + 0.175\alpha_4 &= 0\\ 0.20\alpha_1 + 0.30\alpha_2 + 0.35\alpha_3 + (0.375 - X)\alpha_4 &= 0\end{aligned}$$

which equations are solved for α's and X's on the machine

$$W = 1/\sqrt{x} \text{ in thousands of cycles per min.}$$

For the results of this calculation see Fig. 5.

(b) *The Torsional Divergence Problem*

In this case

$$T_1 = qs_1e_1a_1\alpha_1, \quad T_2 = qs_2e_2a_2\alpha_2$$
$$T_3 = qs_3e_3a_3\alpha_3, \quad T_4 = qs_4e_4a_4\alpha_4$$

where

q = dynamic pressure
s = area of section
e = distance from elastic axis to $^1/_4$ chord point
a = slope of the lift curve
$(a_1 = a_2 = a_3 = a_4 = 6$, say.)

where q in lbs. per sq.in. $= 17.8V^2$ and where V is in (thousands of m.p.h.).

$S_1 = 4{,}800$ sq.in. $\quad e_1 = 10$ in.
$S_2 = 3{,}830$ sq.in. $\quad e_2 = 8$ in.
$S_3 = 2{,}880$ sq.in. $\quad e_3 = 6$ in. $= f_3$
$S_4 = 1{,}920$ sq.in. $\quad e_4 = 4$ in. $= f_4$

(symbol f refers to Fig. 9), or

$$T_1 = 5.12 \times 10^6\ V^2\alpha_1$$
$$T_2 = 3.26 \times 10^6\ V^2\alpha_2$$
$$T_3 = 1.84 \times 10^6\ V^2\alpha_3$$
$$T_4 = 0.82 \times 10^6\ V^2\alpha_4$$

Substituting this and setting $X = 1/V^2$ (V in thousands of m.p.h.),

$$(0.256 - X)\alpha_1 + 0.163\alpha_2 + 0.092\alpha_3 + 0.041\alpha_4 = 0$$
$$0.256\alpha_1 + (0.489 - X)\alpha_2 + 0.276\alpha_3 + 0.123\alpha_4 = 0$$
$$0.256\alpha_1 + 0.489\alpha_2 + (0.644 - X)\alpha_3 + 0.287\alpha_4 = 0$$
$$0.256\alpha_1 + 0.489\alpha_2 + 0.644\alpha_3 + (0.615 - X)\alpha_4 = 0$$

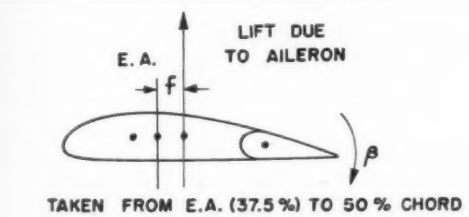
The results of the solution are shown on Fig. 8.

(c) *The Aileron Effectiveness Problem*

In this case we have, in addition to the T's under problem (b), torques due to the aileron in section 3 and 4.

Now the additional lifts are:

$$L_3 = qs_3a_3k\beta, \quad k = {}^1/_2$$
$$L_4 = qs_4a_4k\beta$$
$$T_3 = qs_3a_3kf_3\beta, \quad T_4 = qs_4a_4kf_4\beta$$



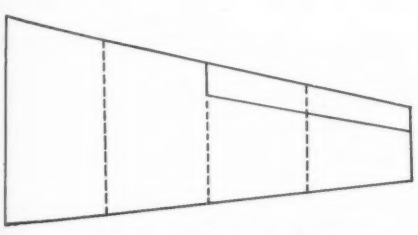


FIG. 9. Wing with aileron.

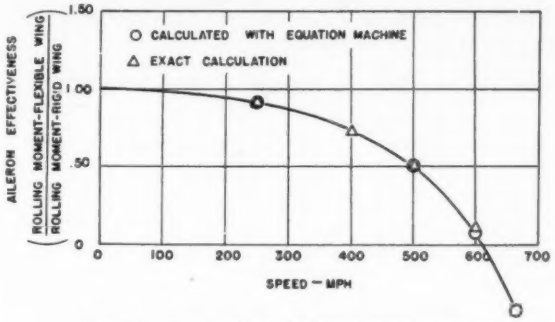


FIG. 10. Comparison of aileron effectiveness.

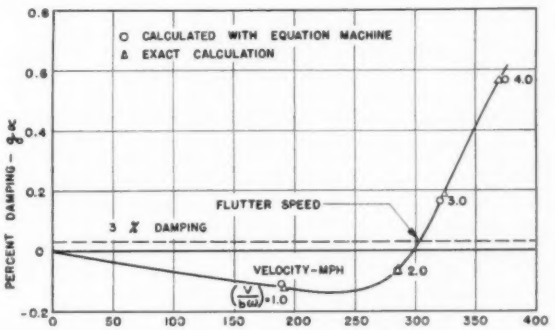


FIG. 11. Comparison of flutter speeds.

Thus, T's are:

$$T_1 = 5.12 \times 10^6\ V^2\alpha_1$$
$$T_2 = 3.26 \times 10^6\ V^2\alpha_2$$
$$T_3 = 1.84 \times 10^6\ V^2\alpha_3 + 0.92 \times 10^6\ V^2\beta$$
$$T_4 = 0.82 \times 10^6\ V^2\alpha_4 + 0.41 \times 10^6\ V^2\beta$$

and the lifts:

$$L_1 = 5.12 \times 10^5\ V^2\alpha_1$$
$$L_2 = 4.17 \times 10^5\ V^2\alpha_2$$
$$L_3 = 3.07 \times 10^5\ V^2\alpha_3 + 1.53 \times 10^5\ V^2\beta$$
$$L_4 = 2.05 \times 10^5\ V^2\alpha_4 + 1.02 \times 10^5\ V^2\beta$$

and the rolling moment:

$$M = 30\ L_1 + 90\ L_2 + 150\ L_3 + 210\ L_4$$

Neglecting wing twist, we would have a moment

$$M_0 = 150L_3 + 210L_4$$

Thus,

$$\frac{M}{M_0} = 1 + 0.347\ \frac{\alpha_1}{\beta} + 0.835\ \frac{\alpha_2}{\beta} + 1.04\ \frac{\alpha_3}{\beta} + 0.97\ \frac{\alpha_4}{\beta}$$

and

$$\frac{\Delta M}{M_0} = \text{per cent increase of rolling moment due to wing twist (actually a decrease)} = 0.347\ (\alpha_1/\beta) + 0.835\ (\alpha_2/\beta) + 1.04\ (\alpha_3/\beta) + 0.97\ (\alpha_4/\beta)$$

Substituting for the T's, we obtain equations similar to that for torsional divergence:

$$[0.256 - (1/V^2)]\alpha_1 + 0.163\alpha_2 + 0.092\alpha_3 + 0.041\alpha_4 = 0.067\,\beta$$
$$0.256\alpha_1 + [0.489 - (1/V^2)]\alpha_2 + 0.276\alpha_3 + 0.123\alpha_4 = 0.200\,\beta$$
$$0.256\alpha_1 + 0.489\alpha_2 + [0.644 - (1/V^2)\alpha_3 + 0.287\alpha_4 = 0.465\,\beta$$
$$0.256\alpha_1 + 0.489\alpha_2 + 0.644\alpha_3 + [0.615 - (1/V^2)]\alpha_4 = 0.630\,\beta$$

(V in 1,000's of m.p.h.). When these equations are solved for the α's, the decrease in rolling moment is given by:

$$\frac{\Delta M}{M_0} = 0.347\left(\frac{\alpha_1}{\beta}\right)0.835\left(\frac{\alpha_2}{\beta}\right)1.04\left(\frac{\alpha_3}{\beta}\right)0.97\left(\frac{\alpha_4}{\beta}\right)$$

To find the *speed* of aileron reversal, we put $\Delta M/M_0 = -1$ and have $\beta = -(0.347\alpha_1 + 0.835\alpha_2 + 1.04\alpha_3 + 0.97\alpha_4)$
and substitute in the above, obtaining an equation similar to that for torsional divergence.

Results of the computations are given on Fig. 10.

(d) *The Flutter Problem*

The flutter problem is of a more complicated character than the problems treated previously, since amplitudes and phases of both bending and torsion of the wing are involved.

We may, however, for purposes of illustration, employ a "typical" section of the wing. In this case, we have only two degrees of freedom, bending and torsion; but, because the motions are not in phase, each degree of freedom is represented by a complex number instead of a real number.

This situation can be handled on a four-equation machine by using one equation for the in-phase (real, say) forces and one for the out-of-phase (imaginary) forces in each degree of freedom. In substance, a two degree of freedom complex system is replaced by a four degree of freedom real system.

Now the equations of equilibrium for a fluttering wing may be written:

$$\left\{\frac{[1 + (A_{hh}/M)]}{(\omega_h/\omega_\alpha)^2} - X\right\}h + \left[\frac{X_\alpha + (A_{h\alpha}/\mu)}{(\omega_h/\omega_\alpha)^2}\right]\alpha = 0$$

$$\left(\frac{X_\alpha}{V_\alpha^2} + \frac{A_{\alpha h}}{\mu V_\alpha^2}\right)h + \left[1 + \frac{A_{\alpha\alpha}}{\mu V\alpha^2} - X(1 + jg_\alpha)\right]\alpha = 0$$

where

X = $(\omega_\alpha^2/\omega^2)$, the square of the ratios of the torsion frequency to the flutter frequency

(ω_h/ω_α) = the ratios of the bending frequency to the torsion frequency

μ = ratio of the wing weight to that of a cylinder of air of diameter equal to the chord

X_α = distance from elastic axis to center of gravity in terms of the half chord

V_α = radius of gyration in terms of the half chord

h = dimensionless bending amplitude (complex)

α = torsion amplitude (complex)

j = $\sqrt{-1}$

g_α = torsion structural damping constant

and the A's represent aerodynamic force coefficients that are complex functions of the reduced velocity $(V/b\omega)$, where V is air velocity, b is the half-chord, and ω is the flutter frequency.

In the following example, we put $\mu = 10$, $(a_h/\omega\alpha) = 1 \div \sqrt{10}$, $X_\alpha = 1/5$, $V\alpha = 1/\sqrt{5}$, so that the equations become

$$(10 + A_{hh} - X)h + (2 + A_{h\alpha})\alpha = 0$$

$$\left(1 + \frac{A_{h\alpha}}{2}\right)h + \left[1 + \frac{A_{\alpha\alpha}}{2} - X(1 + jg_\alpha)\right]\alpha = 0$$

Since the A's depend on $(V/b\omega)$ in a rather complicated fashion, the equations will be solved for various assumed values of this quantity.* A table of the A's is shown here as Table 1.

Now these two complex equations may be put on the machine as four real equations. For example, with $(V/bw) = 1.00$,

$$(10.80 - x)h^R + 1.08h^J + 1.00\alpha^R + 1.58\alpha^J = 0$$
$$(10.80 - x)h^J - 1.08h^R + 1.00\alpha^J - 1.58\alpha^R = 0$$
$$1.14h^R - 0.14h^J + (1.25 - X)\alpha^R + (0.28 + Xg_\alpha)\alpha^I = 0$$
$$1.14h^J + 0.14h^R + (1.25 - X)\alpha^J - (0.28 + Xg_\alpha)\alpha^R = 0$$

where $h = h^R + jh^J = \alpha^R + j\alpha^J$.

Some further modifications are desirable to adapt the problem to the machine. For $(V/b\omega) = 1$ and 2, the equations are divided by 6 and a new "X" term, $\bar{X} = (X - 5)/6$, is used.

Thus, for example, the equations for $(V/b\omega) = 1$ are

$$(0.97 - \bar{X})h^R + 0.18h^J + 0.17 = 0$$
$$-0.18h^R + (0.97 - \bar{X})h^J - 0.26 = 0$$
$$0.19h^R - 0.02h^J - 0.63 - \bar{X} = 0$$
$$0.02h^R + 0.19h^J - 0.05\ (-Xg_\alpha/6) = 0$$

In the above α has been taken as $1 + j0$, since one of the amplitudes is arbitrary.

For the cases $(V/b\omega) = 3$ and 4, the first equation was divided by 50, the second by 10, and the first column was multiplied by 5. $\bar{X}$ was taken as $(X-1)/10$.

* These are for an assumed elastic axis at 38.6 per cent chord to fit conveniently available aerodynamic terms.

TABLE 1

$(V/b\omega)$	A_{hh}	$A_{h\alpha}$	$A_{\alpha h}$	$A_{\alpha\alpha}$
1	0.80–1.08j	−1.00–1.58j	0.28 + 0.28j	0.50 − 0.56j
2	0.40–2.39j	−4.99–2.53j	0.20 + 0.64j	1.60 − 1.30j
3	−0.05–3.90j	−12.23–2.69j	0.50 + 1.06j	3.56 − 2.26j
4	−0.48–5.54j	−23.01–2.11j	0.64 + 1.50j	6.50 − 3.42j

TABLE 2
Results for the Flutter Problem

$(V/b\omega)$	h	$\bar{X}$	$\bar{Y}$	X	g_α	V, M.P.H.
1	−0.13 + 0.13j	−0.65	+0.02	1.1	−0.11	192
2	+0.23 + 0.33j	−0.52	0.02	1.88	−0.064	285
3	0.95 + 0.75j	+0.265	−0.06	3.65	+0.164	322
4	1.65 + 2.25j	+0.38	−0.27	4.8	+0.562	375

Where $b = 30$ in.; $\omega_\alpha = 1{,}125$ cycles per min.; $V = 5.95 \times 30 \dfrac{1.125 V/b\omega}{\sqrt{X}}$. (See also Fig. 11.)

These equations are then solved in the machine for h, $\bar{X}$, and a quantity set in the last constant dial, proportional to $-Xg_\alpha$, and called Y in Table 2.

Accuracy Improvement

The accuracy of a machine solution may be extended to any degree desired by calculating the errors exactly and solving for the increments in the unknowns to an enlarged scale. Suppose, for example, that the equations are written in matrix form $AX = B$, where A is a square matrix of the coefficients A_{ij} and X and B are column matrices of the unknowns and constants. If

$$X = X_1 + \Delta X$$

where X_1 is an approximate solution, then the column matrix of the errors e will be given by

$$AX_1 - B = e$$

and since, by definition,

$$A(X_1 + \Delta X) - B = 0, \quad A\,\Delta X = -e\,.$$

Now the e's and ΔX's will be small so that the scale may be changed to use the full machine capacity. It should be noted that the setting of the coefficients in the machine need not be altered in this process; all that is required is an accurate computation of the errors.

The above process may be applied also to matrices involving a λ term. Thus, if λ_1 and X_1 are approximations to a solution of $(A - \lambda I)X = 0$, where I is the unit matrix, then

$$[A - (\lambda_1 + \Delta\lambda)I](X_1 + \Delta X) = 0$$

or

$$(A - \lambda_1 I)X_1 + (A - \lambda_1 I)\,\Delta X - \Delta\lambda X_1 - \Delta\lambda\,\Delta X = 0$$

Now $(A - \lambda_1 I)X_1 = e$, the matrix of the errors, and if these are computed exactly and if, furthermore, the second order error $\Delta\lambda\,\Delta X$ is neglected,

$$(A - \lambda_1 I)\,\Delta X - \Delta\lambda X_1 = -e$$

Now as one of the X's is arbitrary, the corresponding error may be taken to be exactly zero, so that the n unknown increments will be

$$\Delta X_1,\ \Delta X_2 \ldots .\ \Delta X_{n-1},\ \Delta\lambda$$

Thus, if the last column of the A matrix is replaced by the column matrix X_1 and the matrix of constants (originally set zero) is replaced by $-e$, a solution for the ΔX's and the $\Delta\lambda$ is obtained by solving the above ordinary simultaneous equations to an enlarged scale. If an extra column of coefficients were provided, the $\Delta\lambda X_1$ could be set in without changing the $(A - \lambda I)$ matrix.

In summary, by means of the above methods, a solution of any desired degree of accuracy may be obtained. For example, if the original solution were accurate to two figures, the first improvement would be good to four figures, the next to six, etc.

This method of accuracy extension also provides an independent check of the calculations, including machine errors, dial setting, and reading errors.

It is possible to devise systems of equations which are nearly singular, and any method of computation will have difficulty with these. Experience to date has indicated that engineering problems do not, in general, fall in this class. If a problem is near a singularity, it is sufficient to know this, since any solution that depends on a critical combination of parameters is usually not satisfactory from an engineering standpoint.

Timesaving

The sample problems worked above were done both on the simultaneous equation machine and on desk calculators with experienced operators. A timesaving of 3 to 1 was shown by the equation machine. It is believed that this will be even more favorable on larger order problems. The labor involved in solution on the

(*Continued on page 127*)

A Note on the Approximate Plane Motion During the Burning Period of Rocket-Propelled Missiles Launched at Small Angles of Yaw from Aircraft*

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SUMMARY

The differential equations of plane motion during the burning period of a rocket-propelled missile launched from a moving airplane are written, simplified for approximate application to small angles of yaw (20° or less), and solved. In writing the equations of motion, the gravitational force was neglected, as well as the influence of any pitching disturbances on the yawing forces or moments.

The solution determines the approximate motion of the center of gravity and the spatial orientation of the missile axis as a function of missile design, the initial launching conditions, and the distance along the trajectory. The results are plotted in generalized coordinates, and the curves may then be used for the solution of specific cases.

INTRODUCTION

WHEN A ROCKET-PROPELLED MISSILE is launched from a moving airplane in any direction other than in the direction of flight, the missile is subjected to the aerodynamic forces arising from the resultant angle of yaw. This angle of yaw, which is the angle between the missile axis and the relative wind vector, has in the general case a magnitude within the range of 0° to 180° depending upon the angle of launching, the aircraft velocity, and the missile velocity at launching.

In this paper, the differential equations of motion during the burning period of a rocket-propelled missile launched from a moving airplane are written, simplified for the specific approximate application to small angles of yaw (say 20° or less), and solved. In writing the equations of motion the gravitational force, the influence of any pitching disturbances on the yawing, and the change in mass of the missile during the burning period are all neglected, and all motion is assumed to take place in the plane determined by the vectors of airplane velocity and missile velocity.

Other authors—Peters, Kent, Galbraith, and Sterne at the Aberdeen Research Laboratories; Blitzer at California Institute of Technology; Rosser, Newton, and Gross at the Allegany Ballistics Laboratory,[1] and others—have attacked the problem in various ways. This paper merely presents a somewhat different and rather simple solution of an approximate form of the equations of motion.

The solution derived herein determines the motion of the center of gravity and the spatial orientation of the missile axis as a function of missile design, the initial launching conditions, and the distance (or time) along the trajectory, and it is approximately valid for the range of yaw angles up to about 20°. The results are plotted in generalized coordinates and the curves may then be used for the solution of specific cases.

SYMBOLS

a	= constant, $(K/2m) + (C_1/2I)$
b	= constant, $\{-(C/I) - [(K/2m) - (C_1/2I)]^2\}^{1/2}$
c	= wing chord
C	= constant, $(dC_m/d\alpha)\ (\rho Sc/2)$ (negative for stable missiles)
C_1	= constant, $[dC_m/d(\dot{\alpha}c/2V)]\,(\rho Sc^2/4)$ (always positive)
C_L, C_m	= aerodynamic coefficients of lift and moment
D	= drag
F	= force
K	= constant, $(dC_L/d\alpha)\ (\rho S/2)$
l	= bz
L	= lift
M	= moment
m	= mass
S	= characteristic area, usually wing area
T	= thrust
t	= time
V	= scalar velocity of missile
V_0	= launching velocity of missile with respect to ground
$\boldsymbol{V}$	= vector velocity of missile
z	= distance along trajectory
α	= angle of yaw, between missile axis and relative wind
β	= angle between missile axis and y-axis
γ	= constant of integration, $\gamma = V_0^2 m/2T$
λ	= $b\gamma$
ϕ	= angle between relative velocity vector and y-axis

EQUATIONS OF MOTION

Consider a coordinate system in which the orthogonal axes x and y are located in space in the plane determined by the vectors of the airplane and missile velocities such

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that the y-axis is parallel to the airplane velocity vector at the instant of firing. As shown in Fig. 1, let V be the velocity of the center of gravity of the missile and the vector $\boldsymbol{V}$ is then tangent to the trajectory at every instant. Let ϕ be the angle between $\boldsymbol{V}$ and the y-axis, β the angle between the missile axis and the y-axis, and α the angle of yaw. All angles are taken positive in the clockwise direction.

Now since the gravity forces are neglected the motion of the missile will take place in the x-y plane. The equations of motion are then written with respect to axes fixed in space and the linear motion is resolved along axes parallel and normal to the relative velocity vector at every instant.

The equations of motion are:

$$m\dot{\boldsymbol{V}} = \Sigma \boldsymbol{F} \tag{1a}$$

$$I\ddot{\beta} = \Sigma M \tag{1b}$$

The boldface italicized symbols signify a vector.

Since $\boldsymbol{V} = \boldsymbol{v}_t V$, then

$$\dot{\boldsymbol{V}} = V\dot{\boldsymbol{v}}_t + \dot{V}\boldsymbol{v}_t = V\dot{\phi}\boldsymbol{v}_n + \dot{V}\boldsymbol{v}_t \tag{2}$$

where $\boldsymbol{v}_t$ and $\boldsymbol{v}_n$ are unit vectors parallel and normal to the vector $\boldsymbol{V}$.

Eqs. (1) now become:

$$m\dot{V} = T\cos\alpha - D \tag{3a}$$

$$mV\dot{} = T\sin\alpha + L \tag{3b}$$

$$I\ddot{\beta} = M \tag{3c}$$

If the equations are restricted to small angles of yaw, the two following assumptions may be made:

(1) $\sin\alpha \cong \alpha$; $\cos\alpha \cong 1$.

(2) The drag is negligible compared with T.

The lift, L, and moment, M, are written as:

$$L = KV^2\alpha$$
$$M = CV^2\alpha - C_1V\dot{\beta}$$

where the coefficients

$$K = (dC_L/d\alpha)\ (\rho S/2)$$
$$C = (dC_m/d\alpha)\ (\rho Sc/2)$$
$$C_1 = -\{[dC_m/d(\dot{\beta}c/2V)]\}\ (\rho Sc^2/4)$$

are considered as constants. It will be noted that in the equations above, K and C_1 are always positive numbers whereas C is negative for a statically stable missile and positive for an unstable one.

The approximate equations of motion finally may be written:

$$m\dot{V} = T \tag{4a}$$

$$mV\dot{\phi} = (T + KV^2)\alpha \tag{4b}$$

$$I(\ddot{\phi} + \ddot{\alpha}) = CV^2\alpha - C_1V(\dot{\phi} + \dot{\alpha}) \tag{4c}$$

where $\beta = \phi + \alpha$ and a dot over the symbol signifies differentiation with respect to time

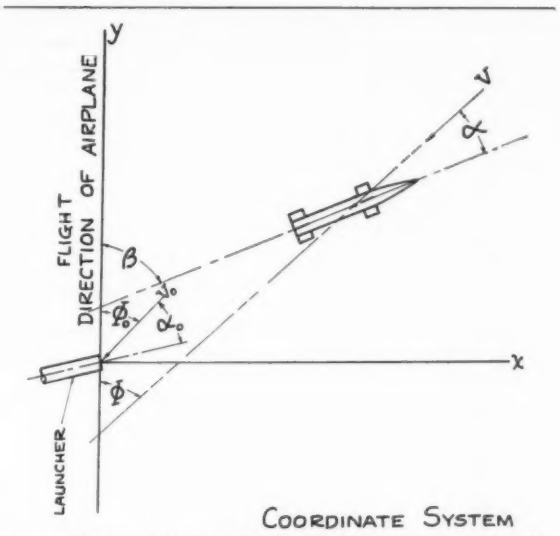


FIG. 1. Schematic diagram illustrating the geometry used for the analysis.

SOLUTION OF EQUATIONS OF MOTION

Eq. (4b) divided by V and differentiated gives:

$$m\ddot{\phi} = (T + KV^2)[(\dot{\alpha}/V) - (\alpha\dot{V}/V^2)] + 2K\dot{V}\alpha \tag{5}$$

Substitution of Eqs. (5) and (4b) into Eq. (4c) gives:

$$\frac{I}{m}\left(\frac{\dot{\alpha}T}{V} - \frac{\alpha\dot{V}T}{V^2} + KV\dot{\alpha} + K\alpha\dot{V}\right) + I\ddot{\alpha} = CV^2\alpha - C_1V\left[\dot{\alpha} + \left(\frac{T + KV^2}{mV}\right)\alpha\right] \tag{6}$$

From Eq. (4a)

$$\dot{V} = T/m \tag{7}$$

and

$$\frac{d}{dt}\left(\frac{V^2}{2}\right) = \frac{T}{m}V = \frac{T}{m}\frac{dz}{dt}$$

$$V^2 = (2T/m)\ (z + \gamma) \tag{8}$$

where

$$\gamma = L_0 + \mu = V_0^2m/2T \tag{8a}$$

Here L_0 is the length of the launcher and μ is a constant to account for the part of the launching velocity of the missile contributed by the airplane velocity. That is, at $z = 0$,

$$V_0^2 = (2T/m)\gamma \tag{9}$$

The independent variable, t, is now changed to the distance, z, along the trajectory of the missile.

Then in the usual manner

$$d/dt = V\ (d/dz) \tag{10}$$

and

$$d^2/dt^2 = V^2(d^2/dz^2) + V\ (dV/dz)\ (d/dz) \tag{11}$$

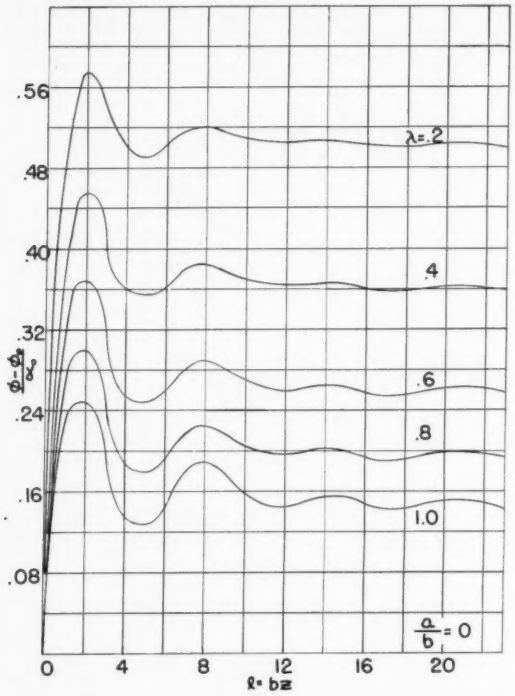


FIG. 2. The dimensionless quantity $\phi - \phi_0/\alpha_0$ (proportional to the change in the angle between the missile velocity vector and the y-axis) plotted as a function of the dimensionless number $l = bz$ (proportional to the distance along the trajectory) for the value $a/b = 0$ and various values of λ. Here a/b is a number dependent upon missile design and λ is a number dependent upon the initial launching conditions.

The use of Eqs. (7), (8), and (10) with Eq. (6) results in

$$\alpha'' + \alpha'\left(\frac{1}{z+\gamma} + \frac{K}{m} + \frac{C_1}{I}\right) + \alpha\left(\frac{K}{2m(z+\gamma)} - \frac{1}{4(z+\gamma)^2} - \frac{C}{I} + \frac{C_1}{2I(z+\gamma)} + \frac{C_1K}{Im}\right) = 0 \quad (12)$$

where $\alpha' = d\alpha/dz$ and $\alpha'' = d^2\alpha/dz^2$. From Eqs. (4b) and (8) then,

$$\phi' = \{[1/2(z+\gamma)] + (K/m)\}\alpha \quad (13)$$

Now let $z + \gamma = X_1$ in Eq. (12). Then,

$$\alpha'' + \alpha'\left(\frac{1}{X_1} + \frac{K}{m} + \frac{C_1}{I}\right) + \alpha\left(\frac{K}{2mX_1} - \frac{1}{4X_1^2} - \frac{C}{I} + \frac{C_1}{2IX_1} + \frac{C_1K}{Im}\right) = 0 \quad (14)$$

If

$$\alpha = e^{-aX_1}X_1^{-1/2}\cos bX_1^* \quad (15)$$

is the assumed form of the solution, an insertion of Eq. (15) into Eq. (14) simplifies to the following expression:

$$a^2X_1^{-1/2}\cos bX_1 + 2abX_1^{-1/2}\sin bX_1 - b^2X_1^{-1/2} \times \cos bX_1 - \frac{aK}{m}X_1^{-1/2}\cos bX_1 - \frac{bK}{m}X_1^{-1/2}\sin bX_1 - \frac{C_1a}{I}X_1^{-1/2}\cos bX_1 - \frac{bC_1}{I}X_1^{-1/2}\sin bX_1 - \frac{C}{I}X_1^{-1/2}\cos bX_1 + \frac{C_1K}{Im}X_1^{-1/2}\cos bX_1 = 0 \quad (16)$$

If the two terms involving the sin bX_1 function are eliminated by proper choice of a, the remaining equation can be solved for b. Obviously a must equal

$$a = K/2m + C_1/2I \quad (17)$$

Eq. (16) becomes

$$b = \sqrt{-(C/I) - [(K/2m) - (C_1/2I)]^2} \quad (18)$$

The constant C is negative for a statically stable missile and an inspection of b shows that $C/I \gg [(K/2m) - (C_1/2I)]^2$ for usual designs. The solution, however, breaks down for a missile designed for nearly neutral static stability or instability.

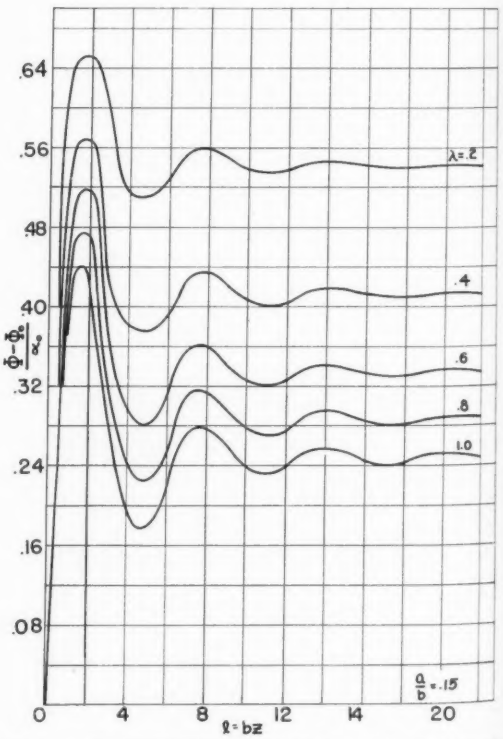


FIG. 3. The dimensionless quantity $\phi - \phi_0/\alpha_0$ as a function of the dimensionless number $l = bz$ for the value $a/b = 0.15$ and various values of λ.

* The general form of this solution was suggested by Dr. D. P. Ling, of the Bell Telephone Laboratories.

Thus Eq. (15) is a solution of Eq. (12) for values of a and b defined by Eqs. (17) and (18). Similarly

$$\alpha = e^{-ax_1} X_1^{-1/2} \sin bX_1$$

is also a solution and therefore the general solution may be written:

$$\alpha = e^{-a(z+\gamma)} (z+\gamma)^{-1/2} \{A \sin [b(z+\gamma)] + B \cos [b(z+\gamma)]\} \quad (19)$$

From Eq. (13)

$$\phi' = \{[1/2(z+\gamma)] + (K/m)\}\alpha$$

Evaluation of the Constants A and B from Boundary Conditions

At the exit of the launcher, $z = 0$, the missile has no yawing angular velocity.

At $z = 0$

$$\begin{aligned} \beta' &= \beta_0' = 0 \\ \alpha &= \alpha_0 \\ \phi &= \phi_0 \end{aligned}$$

Since

$$\begin{aligned} \beta &= \phi + \alpha \\ \phi_0' &= -\alpha_0' \end{aligned}$$

From Eq. (13)

$$\phi_0' = [(1/2\gamma) + (K/m)]\, \alpha_0 = -\alpha_0' \quad (20)$$

And from Eq. (19)

$$\alpha_0' = (-ae^{-a\gamma}\gamma^{-1/2} - (1/2)e^{-a\gamma}\gamma^{-3/2})(A \sin b\gamma + B \cos b\gamma) + e^{-a\gamma}\gamma^{-1/2}(Ab \cos b\gamma - Bb \sin b\gamma) \quad (21)$$

And also

$$\alpha_0 = e^{-a\gamma}\gamma^{-1/2}(A \sin b\gamma + B \cos b\gamma) \quad (22)$$

Combination of Eqs. (20), (21), and (22) results in the evaluation of A and B as:

$$\left.\begin{aligned} A &= (\alpha_0/b)\, e^{a\gamma}\gamma^{1/2}(b \sin b\gamma - a \cos b\gamma) \\ B &= (\alpha_0/b)\, e^{a\gamma}\gamma^{1/2}(b \cos b\gamma + a \sin b\gamma) \end{aligned}\right\} \quad (23)$$

Solution for Angle ϕ

Returning to Eqs. (13) and (19)

$$\phi' = \{[1/2(z+\gamma)] + (K/m)\}e^{-a(z+\gamma)}(z+\gamma)^{-1/2} \times \{A \sin [b(z+\gamma)] + B \cos [b(z+\gamma)]\} \quad (24)$$

Integrating from zero to z,

$$\phi = \phi_0 + 2a\{A \int_0^z e^{-a(z+\gamma)}(z+\gamma)^{-1/2} \times \sin [b(z+\gamma)]\, dz + B \int_0^z e^{-a(z+\gamma)}(z+\gamma)^{-1/2} \times \cos [b(z+\gamma)]dz\} + (1/2)\{A \int_0^z e^{-a(z+\gamma)} \times (z+\gamma)^{-3/2} \sin [b(z+\gamma)]dz + B \int_0^z e^{-a(z+\gamma)} \times (z+\gamma)^{-3/2} \cos [b(z+\gamma)]dz\} \quad (25)$$

If the last two terms are integrated by parts once and combined with the first terms, the equation for ϕ becomes

$$\phi - \phi_0 = \alpha_0 - e^{-a(z+\gamma)}\, (z+\gamma)^{-1/2} \times \{A \sin [b(z+\gamma)] + B \cos [b(z+\gamma)]\} + I_s(aA - bB) + I_c(aB + bA) \quad (26)$$

where A and B are defined by Eq. (23) and

$$\left.\begin{aligned} I_s &= \int_0^z e^{-a(z+\gamma)}\, (z+\gamma)^{-1/2} \sin [b(z+\gamma)]dz \\ I_c &= \int_0^z e^{-a(z+\gamma)}\, (z+\gamma)^{-1/2} \cos [b(z+\gamma)]dz \end{aligned}\right\} \quad (27)$$

Now let

$$\left.\begin{aligned} bz &= l \\ \text{and} \\ b\gamma &= \lambda \\ \text{then} \\ dz &= dl/b \end{aligned}\right\} \quad (28)$$

If Eqs. (28) are inserted into Eqs. (26) and (27) and α_0 factored out:

$$(\phi - \phi_0/\alpha_0) = 1 - e^{-a/b(l+\lambda)}(l+\lambda)^{-1/2} \times [A' \sin (l+\lambda) + B' \cos (l+\lambda)] + I_s[(a/b)\, A' - B'] + I_c[(a/b)\, B' + A'] \quad (29)$$

where

$$\left.\begin{aligned} A' &= e^{(a/b)\lambda}\lambda^{1/2}\, [\sin \lambda - (a/b) \cos \lambda] \\ B' &= e^{(a/b)\lambda}\lambda^{1/2}\, [\cos \lambda + (a/b) \sin \lambda] \end{aligned}\right\} \quad (30)$$

and

$$\left.\begin{aligned} I_s &= \int_0^l e^{-(a/b)(l+\lambda)}\, (l+\lambda)^{-1/2} \sin (l+\lambda)dl \\ I_c &= \int_0^l e^{-(a/b)(l+\lambda)}\, (l+\lambda)^{-1/2} \cos (l+\lambda)dl \end{aligned}\right\} \quad (31)$$

Thus Eq. (29) shows that the change in the angle of the missile velocity vector $\Delta\phi/\alpha_0$ is a function of three dimensionless variables—namely: $l = bz$, proportional to distance z along the trajectory; $\lambda = b\gamma$, a function of the initial launching conditions; and a/b, a function of missile design.

Here

$$\gamma = L_0 + \mu = V_0^2 m/2T \quad (8a)$$

$$a = (K/2m) + (C_1/2I) \quad (17)$$

$$b = \sqrt{-(C/I) - [(K/2m) - (C_1/2I)]^2} \quad (18)$$

Since $\beta = \phi + \alpha$

$$\frac{\beta - \beta_0}{\alpha_0} = I_s\left[\frac{a}{b}A' - B'\right] + I_c\left[\frac{a}{b}B' + A'\right] \quad (32)$$

Thus $\Delta\beta/\alpha_0$ is similarly dependent upon l, λ, and a/b.

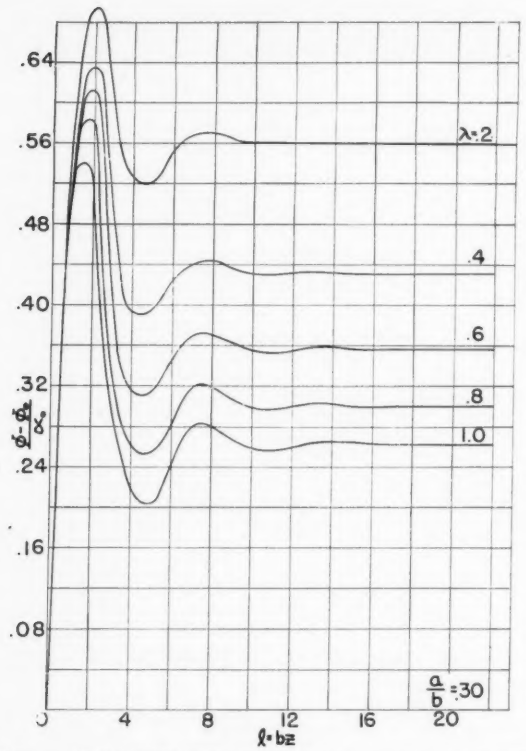


FIG. 4. The dimensionless quantity $\phi - \phi_0/\alpha_0$ as a function of the dimensionless number $l = bz$ for the value $a/b = 0.30$ and various values of λ.

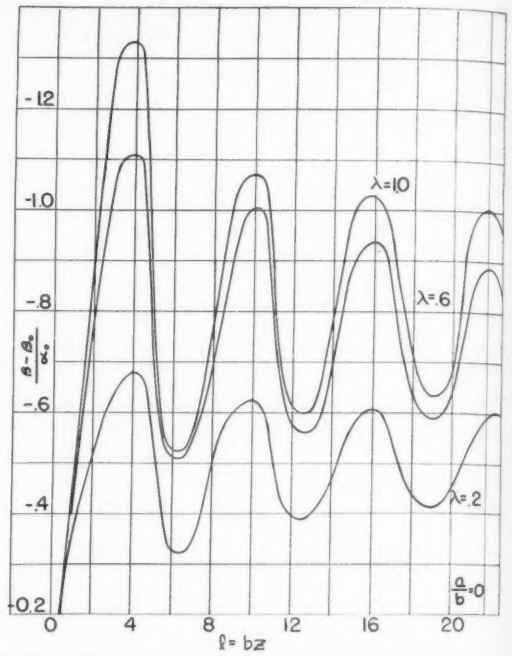


FIG. 5. The dimensionless quantity $\beta - \beta_0/\alpha_0$ (proportional to the change in the angle between the missile axis and the y-axis) plotted as a function of $l = bz$ for the values of $a/b = 0$ and for various values of λ.

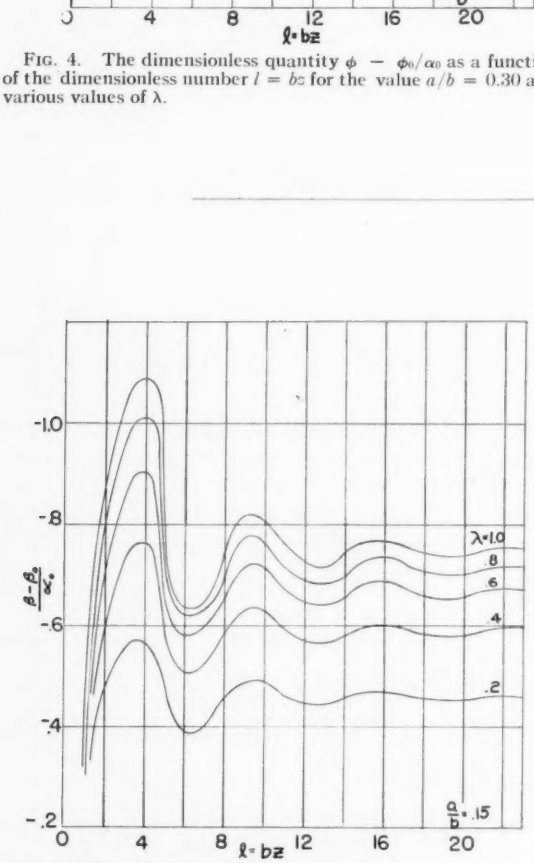


FIG. 6. The dimensionless quantity $\beta - \beta_0/\alpha_0$ as a function of $l = bz$ for the values of $a/b = 0.15$ and for various values of λ.

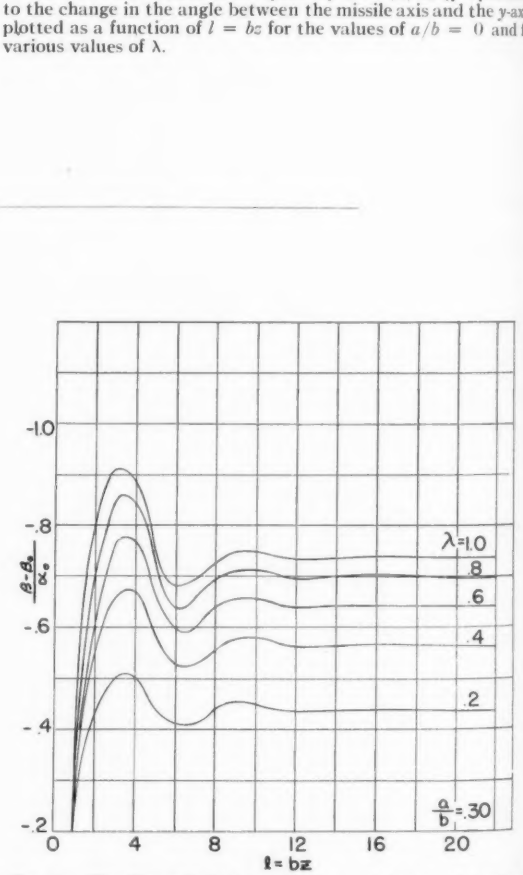


FIG. 7. The dimensionless quantity $\beta - \beta_0/\alpha_0$ as a function of $l = bz$ for the values of $a/b = 0.30$ and for various values of λ.

Discussion

Presentation of the Calculated Results

The dimensionless quantity $(\phi - \phi_0)/\alpha_0$ which is proportional to the change in the angle of the missile velocity vector from its initial value, is plotted in Figs. 2, 3, and 4 against the dimensionless number bz, proportional to the distance along the trajectory, for various values of the dimensionless parameter a/b. Figs. 5, 6, and 7 show the change in the orientation of the missile axis and Figs. 8, 9, and 10 show the respective changes in angle of yaw all plotted in an identical manner. The parameter λ [see Eq. (28)] is a function of the absolute launching velocity of the missile and therefore of the launcher length, airplane velocity, launching direction, and the thrust/mass ratio of the missile. The parameter a/b and the proportionality factor b [Eqs. (17) and (18)] are functions of the missile design only.

The design of a specific missile chosen for study then fixes the values of a/b and b; and the launching velocity (or launching conditions) fixes the value of the parameter λ. The particular $(\phi - \phi_0)/\alpha_0$ curve in Fig. 2, 3, and 4 and the particular $(\beta - \beta_0)/\alpha_0$ curve in Figs. 5, 6, and 7 corresponding to the a/b and λ previously determined show, respectively, the angle of the missile velocity vector (and therefore the slope of the trajectory curve) and the orientation of the missile axis at any point along the missile flight trajectory. The curves presented then give the resultant plane motion during the burning period of *all* missiles included within the scope of the parametric values chosen and for launching conditions which satisfy the approximations made in the equations of motion.

It may be noted from the equations that the wave length of yaw σ is

$$\sigma = 2\pi/b$$

which, for usual missiles where $C/I \gg [(K/2m) - (C_1/2I)]^2$, agrees with the usual definition of σ given in reference 1. The abscissas for the curves, $l = bz$, can then be written

$$l = 2\pi z/\sigma$$

in terms of this wave-length factor.

Approximations

The approximations made during the analysis are (1) the coefficients of force and moment are constant; (2) gravity is neglected; (3) rocket thrust is constant and large compared with the drag forces; (4) the change in mass of the missile during the burning period is neglected; and (5) the angle of yaw is restricted to values where $\sin \alpha = \alpha$ and $\cos \alpha = 1$.

The first approximation cannot be fulfilled when the velocity of the missile attains a very high subsonic or supersonic value except by use of a rough mean value for the coefficients or by a step solution. However, as noted in reference 1, the variations in the coefficients will occur in the latter part of the trajectory (where the high missile velocities are attained) and therefore will have small effect on the trajectory itself. The first approximation also breaks down when the angle of yaw over the initial part of the trajectory is large enough to cause variations in the coefficients. This latter condition was investigated in the Rensselaer Poly-

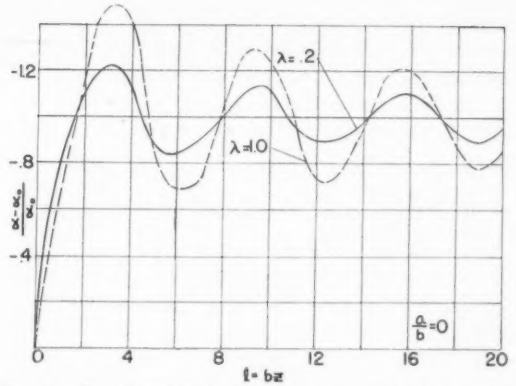


FIG. 8. The dimensionless quantity $\alpha - \alpha_0/\alpha_0$ (proportional to the change in the angle of yaw) as a function of $l = bz$ for the value $a/b = 0$ and various values of λ.

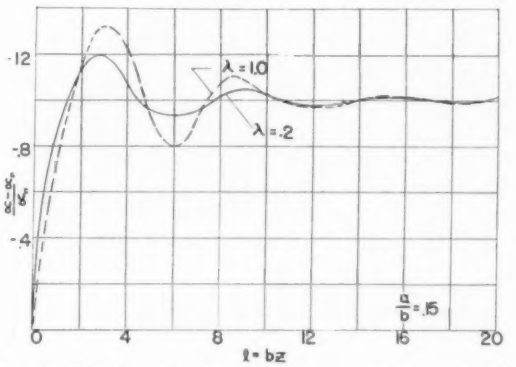


FIG. 9. The dimensionless quantity $\alpha - \alpha_0/\alpha_0$ as a function of $l = bz$ for the value $a/b = 0.15$ and various values of λ.

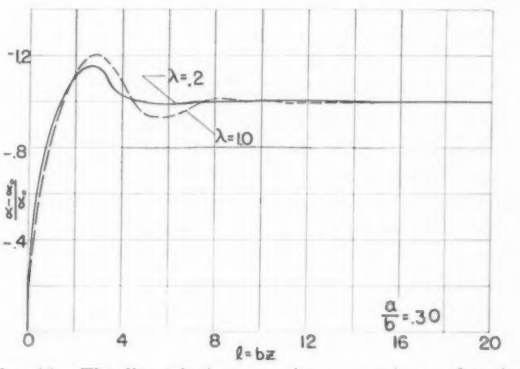


FIG. 10. The dimensionless quantity $\alpha - \alpha_0/\alpha_0$ as a function of $l = bz$ for the value $a/b = 0.30$ and various values of λ.

technic Institute subsonic wind tunnel for a general supersonic missile configuration and the results are shown in Fig. 11. The lift and center of pressure (c.p.) curves show an approximately linear relation for the lift and moment coefficients with angle of yaw up to about 30° or 40°.

Brief Description of the Resulting Motion

The motion of the missile is, of course, a result of the missile design, the wind velocity, and the initial angle of yaw. Upon launching, the moment of force acting on a statically stable missile rotates the missile very quickly toward the direction of the realtive wind vector. At the same time the acceleration of the missile along its own axis resulting from the rocket thrust and the velocity induced by the lift force cause the relative wind vector to swing around toward the missile axis. The relative magnitude of these two effects, as well as of the damping moment, then determines the missile trajectory as well as the orientation of the missile axis throughout the rocket burning period.

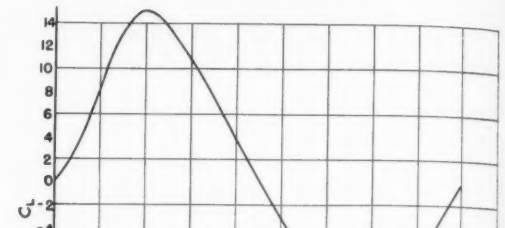
If the missile, however, were very nearly neutrally stable, then almost no rotation of the missile axis would take place and the wind vector would rapidly align itself with the missile axis resulting in little angular deviation from the launching condition.

Discussion of Calculated Results

The values of a/b chosen for the calculations are extreme and were chosen to illustrate the general characteristics of the solution. The value $a/b = 0$ has no physical significance but is of interest as a limiting value only.

From a comparison of the curves and an inspection of Eq. (29) it is seen that the lift coefficient term $(K/2m)$ acts as a damping factor in almost identically the same manner as the damping moment term $(C_l/2I)$ for statically stable missiles where $C/I \gg [(K/2m) - (C_l/2I)]^2$ and thus $b \cong \sqrt{C/I}$. However, for missiles with very small static stability, the value of b is appreciably affected by the value of $[(K/2m) - (C_l/2I)]$ while $a = K/2m + C_l/2I$. Since both a and b are damping factors, the a in the exponential term and the b in the $(l + \lambda)^{-1/2}$ term, the damping action of the lift force differs from that of the damping moment for this particular case.

The physical action of the lift force in producing damping results from the fact that lift induces a velocity component normal to the trajectory in a direction which always aids the relative wind vector to swing around into the path of the missile, thereby decreasing the angle that the missile will rotate in its simultaneous attempt to swing around into the wind vector. Any factor causing a decrease in the angular displacement of the missile appears in the resultant motion as a damping factor.



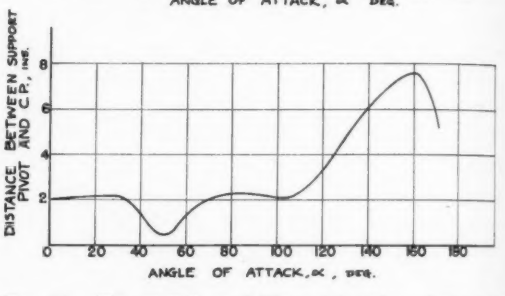


FIG. 11. The coefficient of lift and location of the missile center of pressure (c.p.) measured in the R.P.I. tunnel as a function of angle of yaw from 0° to 180°. The missile consisted of a cone-cylinder fuselage 25 in. in length and with cruciform wings and tail surfaces attached and was tested at 150 ft. per sec. air velocity.

The relative movement of the wind vector and the missile axis is given by a comparison of the $\Delta\phi/\alpha_0$ and the $\Delta B/\alpha_0$ curves for any missile within the range of variables chosen for calculation. The damping effectiveness is observed in the $\Delta\beta/\alpha_0$ curves.

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[1] Rosser, J. B., Newton, R. R., and Gross, G. L., *Mathematical Theory of Rocket Flight;* McGraw-Hill Book Company, Inc., 1947.

Seventy-Fifth Anniversary of Ludwig Prandtl

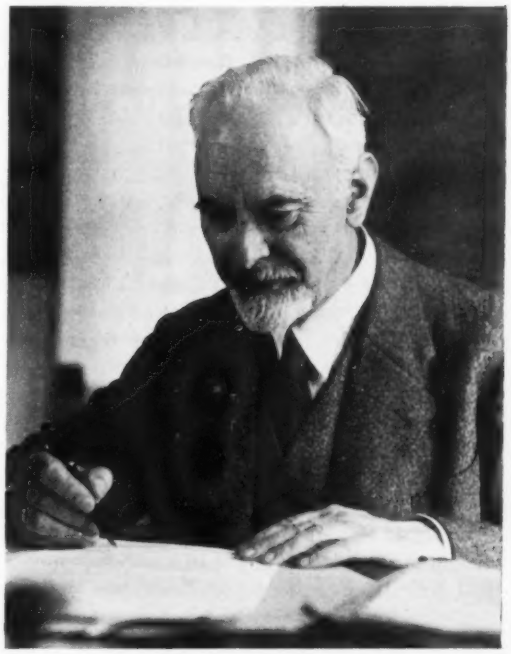
WHEN THE ART OF FLYING began its modern growth in the early days of the century, little was known of what today we call aerodynamics. Thoughtful observation of birds and model experiments with wings played an important part in the minds of daring inventors. Today everyone is aware that the production of airplanes is a major industry. It is well known that, in the rapid development of this new branch of engineering, scientific research played an important part.

One of the outstanding figures in this field is Ludwig Prandtl in Göttingen, who on February 4, 1950, will complete his 75th year. In recognition of his world-wide influence and prominent leadership, a survey of his scientific work is here presented to our readers.

Long before governments recognized the importance of airplanes or expended any funds on their development, Prandtl began his studies of airflow and threw light on one of the darkest corners of applied mechanics. At the International Mathematics Congress in Heidelberg in 1904, he presented the famous paper on the motion of fluids with extremely small viscosity, which laid the foundation of boundary-layer theory. It gave the explanation of the great discrepancy between the theory of potential flow and observed facts and showed how and why this discrepancy could be avoided in cases of aeronautical importance. Much later, when aircraft had developed so far that extreme reduction of drag became a question of vital importance, Prandtl's boundary-layer theory became a subject of world-wide mathematical investigation. This investigation still continues.

Soon after his publication on boundary layers, Prandtl developed an active interest in airships, which led to the creation of Germany's first wind tunnel. The design of this tunnel is visibly influenced by Prandtl's conception of the boundary layer: The free jet avoids all wall friction in the test section, and the convergent collector reduces the inhomogeneity of the air stream because of the boundary layer developed in the duct. The laboratory that developed around this tunnel ranked among the first of its kind and became one of the leading centers of aerodynamic research. Not only did it yield many valuable experimental and theoretical results, but it also turned out a considerable number of excellent research workers who spread all over the world and brought with them that Göttingen spirit of applied research which so happily connects pure science with the practical problems of the engineer.

The second of the great impulses which Prandtl gave to aeronautical science developed in the secrecy of a war period. Published first in restricted form in 1918, then in 1919 in the *Nachrichten der Göttinger Gesellschaft der Wissenschaften*, his papers on airfoil theory soon became the base of research activities in all countries where airplanes are built. It is well known how far the shape of airplanes was changed by this research.



Ludwig Prandtl

Prandtl's publications on both subjects, boundary layer and wing theory, demonstrate his specific way of scientific work. He shows surprising relations, previously overlooked or incompletely understood, gives the basic ideas for their analysis, and so inspires many other workers to continue the mathematical development and work out the technical applications. This has been characteristic of him from his Doctor's thesis to the present day.

In the nineteen-twenties, public interest in aviation increased enormously. The strange flying machines of the prewar period were developing into the new streamlined airplanes. The war having shown beyond doubt their value for destructive purposes, the question of peacetime use presented itself. The days of the pioneers had ended, and commercial aviation began. In these years, Prandtl's position in aeronautical research also changed from that of the scientific trailblazer to that of one of the most prominent men in a world-wide family of scientists. His further publications are no longer explorer's trips into new lands but are leading contributions to questions of great

interest which find an immediate response beyond the circle of his personal followers.

Thus he participates in the investigation of the turbulence problem, seeing its full breadth from the velocity distribution in pipes to the turbulence of the free atmosphere. His particular interest is devoted to the transition from laminar to turbulent flow in pipes and along walls, and he initiates theoretical investigations on the stability of the flow in the boundary layer. The mixing zone on the surface of free jets also attracts his attention, and the notion of the mixing length, formulated by analogy with the free-path length in the kinetic theory of gases, proved to be a new and useful tool in this field.

As early as 1904, Prandtl had published papers on the dynamics of compressible fluids. The topic was resumed in more recent times, when airplanes reached such speed that the compressibility of air became of substantial importance for their performance.

Less known to aeronautical engineers is Prandtl's work outside aerodynamics, and many among the younger generation may not know that he began his scientific life with a Doctor's thesis on a problem of elastic stability. This part of the theory of elasticity, today so vital in aircraft design, was rather undeveloped at the end of the last century. In his thesis, published in 1900, Prandtl presented and solved a *new* problem—that of the lateral instability of beams in bending. More than that, in his experiments that accompanied the theory, he conceived and developed the method of adjusting the test specimen so that the influence of its imperfections on the critical load is compensated—now a standard practice in all careful buckling experiments.

The second of Prandtl's early papers on elastic problems is known to a larger public. In 1904 he discovered the soap-film analogy for the Saint-Venant torsion problem. Because of its great suggestiveness, it became one of the standard topics of textbooks dealing with torsion, and its practical value for stress analysis has been recognized by many engineers.

In 1920 his first paper on the theory of plasticity appeared, and, subsequently, we find the subject appearing at intervals in his publication list in contributions to knowledge which have stimulated interest in the subject in Göttingen and elsewhere.

This sketch of Prandtl's scientific activities, necessarily far from exhaustive, is not enough to describe his influence on present-day applied mechanics. Perhaps an even more important aspect of his activity is that of an academic teacher of the first rank, gathering through almost half a century an ever-changing group of pupils about him. These young men (and not only young ones) acquired from him both scientific knowledge and a scientific attitude. They continue the work so begun in many parts of the world—not a few of them in this country—contributing to the training of a new generation of research workers and engineers. Their thoughts will be with Prandtl on his 75th anniversary, and they will wish to join with fellow scientists the world over in saluting him as one of the greatest figures of engineering science.

Discussion of Wright Brothers Lecture

(*Continued from page 106*)

accurate measurement of the free-body vibration modes. In our own thinking on this problem, we have come to regard the ground vibration test merely as a means for verification of the mass and stiffness distributions and of the method employed to calculate vibration modes. If this philosophy is adopted, then there is no real necessity for simulating the free-body condition, since comparisons may be made between measured and calculated modes for a rigidly supported structure.

Finally, with regard to flight vibration tests, it is contended that the use of a vibration exciter does not in any way increase the hazard of the tests. Of course, it is essential to differentiate between vibration testing and flutter testing in flight. If, as implied by Mr. Russell, the test is conducted solely for the determination of vibration modes and natural frequencies of the airplane in free flight, then it must be assumed that the airplane will not be operated deliberately near a critical flutter speed. If such a deplorable situation should occur, then the vibration exciter at least gives some assurance that the critical speed will not be exceeded by a large margin, thereby reducing the likelihood of catastrophic flutter.

Readers' Forum

BRIEF REPORTS of investigations in the aeronautical sciences and discussions of papers published in the JOURNAL *are presented in this special department. Publication will be completed 6 to 8 weeks after receipt of the material. No proof will be sent to the authors. The Editorial Committee does not hold itself responsible for the opinions expressed by the correspondents. Contributions should not exceed 800 words in length.*

Note on Rotational Gas Flow

James E. Broadwell
Ann Arbor, Mich.
October 10, 1949

SYMBOLS

$\bar{H}$ = angular momentum about center of mass
κ = unit vector perpendicular to plane of flow
$\bar{L}$ = torque about center of mass
m = mass
p = pressure
$\bar{P}$ = vector to center of mass, with rectangular components x_1 and y_1
$\bar{q}$ = velocity, with rectangular components u and v
$\bar{r}$ = distance from origin with rectangular components x and y
S = area
ρ = density

THE LAWS OF conservation of mass and momentum for the two-dimensional, rotational, diabatic flow of a perfect compressible fluid may be put in the form:

$$(\nabla \rho^{-1}) \times (\nabla p) = -\rho \frac{d}{dt}(\rho^{-1}\omega) \qquad (1)^1$$

where

$$\omega \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

and the symbol d/dt indicates differentiation following the motion of the fluid. In the following, Eq. (1) is derived by equating the torque which acts on a fluid particle, assumed to be square, to the rate of change of angular momentum of that particle. Such a derivation would perhaps be useful in teaching fluid mechanics where the concept of rotation causes some difficulty.

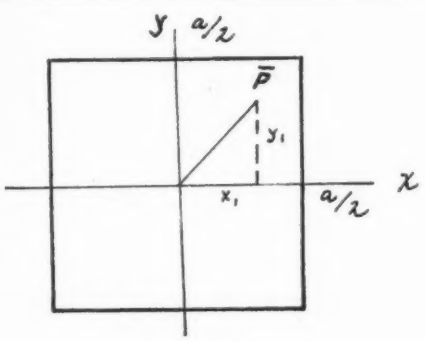


FIG. 1.

Consider a square fluid element of unit thickness at the origin as shown in Fig. 1. For any element of the fluid,

$$\bar{L} = \frac{d\bar{H}}{dt} \qquad (2)$$

where

$$\bar{H} \equiv \kappa \int_S \int [(\bar{r} - \bar{p}) \times \bar{q}]\rho \, dS$$

For the above element,

$$\bar{H} = \kappa \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} [(x - x_1)v - (y - y_1)u]\rho \, d\,x\,dy \qquad (3)$$

Let the particle be of such size that the density and velocities inside are given by the equations

$$\left.\begin{aligned} \rho &= \rho_0 + \left(\frac{\partial \rho}{\partial x}\right)_0 x + \left(\frac{\partial \rho}{\partial y}\right)_0 y \\ u &= u_0 + \left(\frac{\partial u}{\partial x}\right)_0 x + \left(\frac{\partial u}{\partial y}\right)_0 y \end{aligned}\right\} \qquad (4)$$

and likewise for v, where the subscript 0 denotes values at the origin. The center of mass $\bar{P}$ is defined by the equation

$$\bar{P} \equiv \frac{1}{M}\int_S \int \bar{r}\, dm$$

where M is the total mass of the element. With Eqs. (4) the components of $\bar{P}$ become

$$\left.\begin{aligned} x_1 &= \frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial x}\right)_0 \frac{a^2}{12} \\ y_1 &= \frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial y}\right)_0 \frac{a^2}{12} \end{aligned}\right\} \qquad (5)$$

Substitution of Eqs. (4) and (5) in Eq. (3) and integration result in

$$\bar{H} = \kappa\rho_0 a^2 \frac{\omega_0}{2}\frac{a^2}{6} \qquad (6)$$

if terms of order a^4 and higher are dropped. Note that this is the expression for the angular momentum, measured about the center, of a square of uniform density and angular velocity $\omega_0/2$.

The torque, which results only from pressure forces, is given by

$$L = \kappa\left[\int_{-a/2}^{a/2} (y - y_1)p\, dy\right]_{-a/2}^{a/2} - \kappa\left[\int_{-a/2}^{a/2} (x - x_1)p\, dx\right]_{-a/2}^{a/2} \qquad (7)$$

Setting

$$p = p_0 + \left(\frac{\partial p}{\partial x}\right)_0 x + \left(\frac{\partial p}{\partial y}\right)_0 y$$

substituting from Eq. (5) and integrating, changes Eq. (7) to

$$\bar{L} = -\kappa\left[\left(\frac{\partial \rho}{\partial y}\right)_0\left(\frac{\partial p}{\partial x}\right)_0 - \left(\frac{\partial \rho}{\partial x}\right)_0\left(\frac{\partial p}{\partial y}\right)_0\right]\frac{a^4}{12\rho_0} \qquad (8)$$

Eqs. (6) and (8) combine through Eq. (2) to

$$-\left[\left(\frac{\partial \rho}{\partial y}\right)_0\left(\frac{\partial p}{\partial x}\right)_0 - \left(\frac{\partial \rho}{\partial x}\right)_0\left(\frac{\partial p}{\partial y}\right)_0\right]\frac{a^4}{12\rho_0} = \frac{d}{dt}\left[\frac{\rho_0\omega_0 a^4}{12}\right] \qquad (9)$$

Since Eq. (9) is to be applied to a particle as it moves along the streamline, the mass, $\rho_0 a^2$ to the order of Eq. (9), must be held constant. This fact changes Eq. (9) to

$$-\frac{1}{\rho^3}\left[\left(\frac{\partial \rho}{\partial y}\right)_0\left(\frac{\partial p}{\partial x}\right)_0 - \left(\frac{\partial \rho}{\partial x}\right)_0\left(\frac{\partial p}{\partial y}\right)_0\right] = \frac{d}{dt}\left[\rho_0^{-1}\omega_0\right]$$

or, dropping the subscript, to Eq. (1),

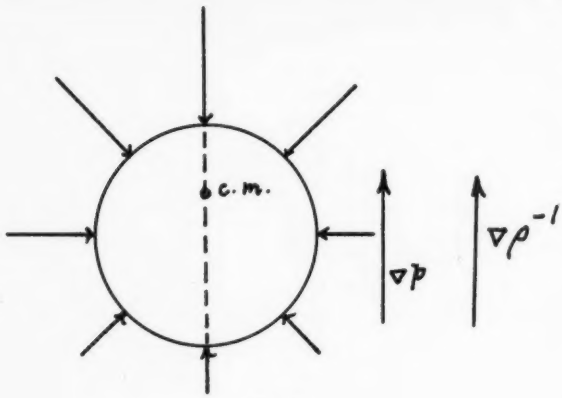


FIG. 2.

$$(\nabla \rho^{-1}) \times (\nabla p) = -\rho \frac{d}{dt} (\rho^{-1}\omega)$$

From the above derivation it is seen that the quantity $(\nabla \rho^{-1}) \times (\nabla p)$ is proportional to the torque acting on the element and that $\omega\rho^{-1}$ is proportional to the angular momentum. In the simple case of a circular disc of compressible matter held at uniform density, ρ, it is easily shown that the angular momentum is proportional to $\Omega\rho^{-1}$ where Ω is the angular velocity. For a circular disc of radius R,

$$H = (1/2)\pi R^2 \rho \Omega$$

If the body expands radially and if the mass, $\rho\pi R^2$, remains constant,

$$H = \frac{1}{2} \frac{(\rho\pi R^2)^2}{\pi} \frac{\Omega}{\rho} = \text{constant} \frac{\Omega}{\rho}$$

The physical reason for the angular momentum remaining constant when ρ is a function only of p (so that $(\nabla \rho^{-1}) \times \nabla p$ is zero) is pointed out by F. H. Clauser in a Douglas Company report.[2] Consider again a circular element in a flow field (see Fig. 2). If ρ is a function p only, then the center of mass lies somewhere on the dotted line parallel to Δp and the torque about this point is zero. This is not the case, of course, if ρ is independent of p as in diabatic flow.

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[1] Vazsonyi, A., *On Rotational Gas Flow*, Quart. of Applied Math., Vol. III, No. 1, April, 1945.

[2] Clauser, F. H., *Theory of Compressible Fluids*, Douglas Aircraft Company Report, April 8, 1944.

Errata—The Aerodynamic Forces on an Oscillating Flap at Supersonic Speeds

John W. Miles
Department of Engineering, University of California at Los Angeles
September 26, 1949

EDWARD TAYLOR of Douglas Aircraft, Inc., has pointed out to us that Eq. (22) and Figs. 8 and 9 in this paper were correct as originally stated and not in error, as stated in a previous errata (JOURNAL OF THE AERONAUTICAL SCIENCES, Vol. 16, No. 7, p. 442, July, 1949).

The Mechanism of Unstable Combustion

W. C. Randels
Northrop Aircraft, Inc., Hawthorne, Calif.
November 4, 1949

THE PHENOMENA OF unsteady burning in forced-draft combustion systems has been frequently observed. The purpose of this note is to show that the possibility of such occurrences might be expected on the basis of a conventional stability analysis.

Consider a system consisting of a compressor, combustion chamber and nozzle as in Fig. 1. The performance of the compressor can be expressed by the "characteristic" curve of pressure as a function of mass flow as Curve A in Fig. 2. Similarly, the total pressure required by the combustion chamber can be graphed, and the intersection with the compressor characteristic gives the operating point. Two types of intersection are illustrated by Curves B and C in Fig. 2. The difference between these types can be explained as follows: Consider Curves A and B in the vicinity of the point of intersection in Fig. 3. If for any reason an increase in flow changes the operating point to Point 1 at some instant, then at that time the pressure supplied by the compressor will be less than that required by the combustion chamber and nozzle, and this pressure deficiency produces a decelerating force on the air column so that the mass flow tends to return to the equilibrium position. Such an intersection is called stable. On the other hand, for an intersection of the type A,C (Fig. 4) a displacement to Point 1 produces an excess of available pressure over that required which applies an accelerating

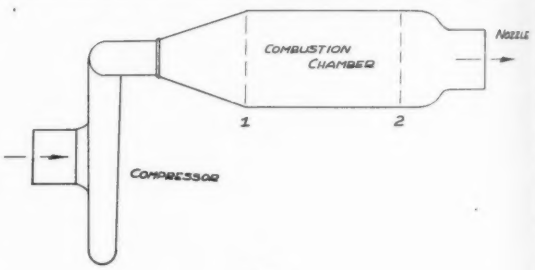

FIG. 1.

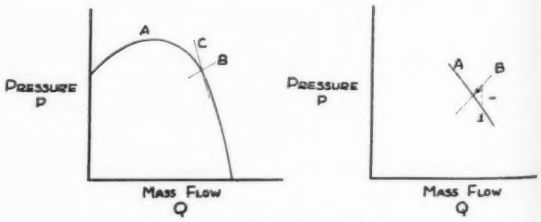

FIG. 2 (*left*). FIG. 3 (*right*).

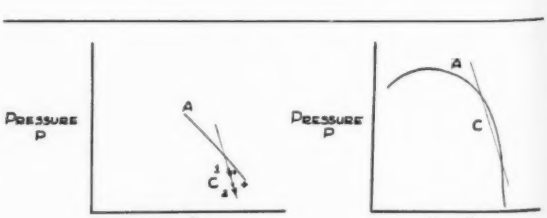

FIG. 4 (*left*). FIG. 5 (*right*).

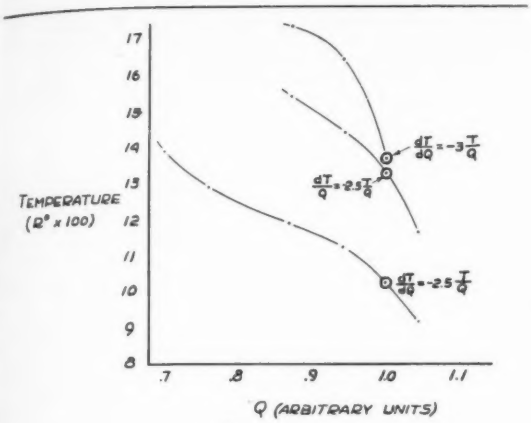

FIG. 6.

force to the air stream which tends to move the operating point to 2, and so on. Such an intersection is called unstable. If Curve C recrosses A, as in Fig. 5, vibration may occur. If it does not recross, the solution diverges steadily, causing blowout.

Since most systems operate on the region of the compressor characteristic with negative slope, it must be shown that it is possible to have pressure-required curves with negative slope. In order to make a simpler problem computationally, pressure losses in the combustion chamber are neglected, and all dynamic pressures ($^1/_2\rho V^2$) are assumed to be small compared to ambient pressure (P_a). Incompressible formulas can then be used. The pressure required by the nozzle is

$$P_1 - P_a = (1/2)\rho_2 V_2^2$$

and, since

$$\rho_2 V_2 A_2 = Q$$

and

$$\rho_2 = (T_1/T_2)\rho_1$$

this can be written

$$P_1 - P_a = (1/2)\,(Q^2/\rho_1 A_2)\,(T_2/T_1)$$

The slope dP_1/dQ, is

$$\frac{dP_1}{dQ} = \frac{1}{2\rho_1 A_2 T_1}\left(2QT_2 + Q^2\frac{dT_2}{dQ}\right) = \frac{1}{2\rho_1 A_1}\frac{Q^2}{T_1}\left(2\frac{T_2}{Q} + \frac{dT_2}{dQ}\right)$$

The quantity in brackets will be negative if the combustion efficiency drops off with velocity enough that

$$-dT_2/dQ > 2(T_2/Q)$$

The data of reference 1 have been analyzed from this point of view. These data were taken from tests of an annular turbojet combustor. This is not too close to the one-dimensional model, since it is possible to have local instability at isolated points without general overall instability. As a matter of fact, three kinds of instability are reported as having occurred separately or in combination. One was observed visually as a flicker in the burning zone. One was observed as fluctuations in the exit temperature. The third was observed as noise and duct vibration. Since the above analysis involves pressure stability, it should apply only to the third case, and so only such points have been studied. In Fig. 6 (data from reference 1, Figs. 8c and 9c) have been cross-plotted for constant fuel flow, and the points of unstable combustion are indicated. Since the compressor characteristics are not given, it can only be said that the agreement with analysis is plausible.

On the basis of this analysis, certain things can be pointed out which can help stabilize combustion. For one thing, operating on the steep (constant flow) portion of the characteristic curve should help stability. For another, a metering system designed for constant heat release would maintain stable combustion if the metering system were fast enough.

REFERENCE

[1] Childs, J. Howard, McCafferty, Richard J., and Surine, Oakley W., *Effect of Combustor-Inlet Conditions on Performance of an Annular Turbojet Combustor*, N.A.C.A. Report No. 881, 1947.

Errata—"Finite Deflections of Sandwich Plates"*

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November 3, 1949

IN THIS PAPER three of the equations should be corrected as follows:

$$\left.\begin{aligned} M_x = (N_{x+} - N_{x-})(h/2), \quad M_y = (N_{y+} - N_{y-})(h/2) \\ H = (S_+ - S_-)(h/2) \end{aligned}\right\} \quad (28)$$

$$p + \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + N_x\frac{\partial^2 w}{\partial x^2} + 2S\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2} + M_x\frac{\partial^2 e}{\partial x^2} + 2H\frac{\partial^2 e}{\partial x \partial y} + M_y\frac{\partial^2 e}{\partial y^2} = 0 \quad (35)$$

$$q - \bar{\sigma}_z + \frac{h}{4}\left(N_x\frac{\partial^2 e}{\partial x^2} + 2S\frac{\partial^2 e}{\partial x \partial y} + N_y\frac{\partial^2 e}{\partial y^2}\right) + \frac{1}{h}\left(M_x\frac{\partial^2 w}{\partial x^2} + 2H\frac{\partial^2 w}{\partial x \partial y} + M_y\frac{\partial^2 w}{\partial y^2}\right) = 0 \quad (36)$$

The error committed was in the nature of an oversight in algebra. I am very much indebted to Prof. C. T. Wang, of New York University, for discovering the necessity of the above corrections. As the terms that have been changed in Eqs. (35) and (36) were not used in the remainder of the paper, none of the subsequent results is affected by these changes.

* JOURNAL OF THE AERONAUTICAL SCIENCES, Vol. 15, No. 7, pp. 435–440, July, 1948.

Detonation and Static Electricity in the Tropical Atmosphere

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October 31, 1949

IN 1941, THE AUTHOR submitted an explanation of the experiments of Corbitt and Clark[1] on aero-engines detonating when electrostatically cleaned air was admitted to the intake, respectively,[2] of experiences of aero-engines detonating during flight in or after thunderstorms in the tropics. This explanation was not published until 1946 and 1947.[2,3] It seemed probable that ozone had been formed in both cases, which induced and/or accelerated chain reactions so that detonation occurred.

Recently, when taking stock of literature that had appeared in Indonesia before the war and which had survived, the author found some figures on measurements of electrostatic charges occurring during tropical rains.[4] In these measurements, de

Kruyff found positive charges up to 2,362 e.g.s. and negative charges up to 2,324 e.g.s. The largest specific charge found was 4.7 e.g.s. per cu.cm.

Small showers had higher specific charges than heavy showers.

References

[1] Corbitt, H. E., and Clark, N. J. *Electrostatic Precipitation for Aircraft*, Aero Digest, Aviation Engineering, p. 132, December, 1940.

[2] Broersma, G., *Detonation*, Natuurwetenschappelijk Tijdschrift, Vol. 102, p. 80, 1946.

[3] Broersma, G., *Detonatie*, Polytechnisch Tijdschrift, p. 519a, 1947.

[4] de Kruyff, J. P. H., *Metingen van electrische lading van regens te Soerabaia*, Handelingen van het Achtste Nederlandsch-Indisch Natuurwetenschappelijk Congres te Soerabaia, 140, July 1938, 20–23.

Second-Order Two-Dimensional Supersonic Airfoil Theory

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October 20, 1949

In determining the higher order correction terms to the linear supersonic airfoil theory of Ackeret, the usual procedure has been to expand either the two-dimensional oblique shock equations or the Prandtl-Meyer equation into a power series in θ, the flow deflection angle.[1] A third method to be presented here consists in replacing the airfoil or wedge by sources producing either isentropic waves of finite amplitude or weak shock waves, thus extending Taylor's result.[2] The flow can then be thought of as being essentially one-dimensional if only outgoing waves are present. The computations will be carried out to terms in θ^2.

First, by utilizing the isentropic relations

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} = \left(\frac{T}{T_0}\right)^{\gamma/\gamma\,-\,1} = \left(\frac{a}{a_0}\right)^{2\gamma/(\gamma\,-\,1)}$$

where a = speed of sound, the one-dimensional equation of continuity and momentum are:

$$\left.\begin{aligned} [(\gamma - 1)/2]\, a u_x + a_t + u a_x &= 0 \\ u_t + u u_x + [2a/(\gamma - 1)] a_x &= 0 \end{aligned}\right\} \tag{1}$$

where γ is the ratio of specific heats and u is the particle velocity. Substituting $a = a(u)$ (which is possible since the motion is supposed isentropic), and multiplying the second equation by da/du and subtracting from the first, we get:

$$da/du = \pm(\gamma - 1)/2 \quad \text{or} \quad a = a_0 \pm [(\gamma - 1)/2]u$$

Considering only compression flows for convenience, the positive sign is appropriate. Hence:

$$P/P_0 = \{1 + [(\gamma - 1)/2]\,(u/a_0)\}^{(2\gamma/\gamma\,-\,1)} \tag{2}$$

The particle velocity must be normal to the wave front and of such magnitude as to equal the component of free-stream velocity normal to the airfoil, $\bar{u}\sin\theta$. Therefore, $u = \bar{u}\sin\theta/\cos\phi = \bar{u}\theta/\cos\phi$, θ small. (See Fig. 1.)

The wave velocity itself is given by:

$$W = \frac{1}{2}\left[a_0 + a_0 + \left(\frac{\gamma + 1}{2}\right)u\right] = a_0 + \left(\frac{\gamma + 1}{4}\right)u$$

so that:

$$\sin\theta_W = \left(a_0 + \frac{\gamma + 1}{4}\,u\right)/\bar{u} = \frac{1}{M} + \frac{\gamma + 1}{4}\left(\frac{\theta}{\cos\phi}\right) \tag{3}$$

By expanding Eq. (2), we find, to terms in θ^2:

$$\frac{P - P_0}{\gamma P_0 M^2}/2 = \frac{\Delta P}{q} = \frac{2\theta}{M\cos\phi} + \frac{\gamma + 1}{2}\left(\frac{\theta^2}{\cos^2\phi}\right) \tag{4}$$

Putting $\theta_W = \sin^{-1}(1/M) + \delta$, where δ is small, into Eq. (3) gives an equation for δ, which to the necessary order of approximation is solved by:

$$\delta = \left(\frac{\gamma + 1}{4}\right)\theta\left(\frac{M^2}{M^2 - 1}\right)$$

$$\theta_W = \sin^{-1}\left(\frac{1}{M}\right) + \left(\frac{\gamma + 1}{4}\right)\theta\left(\frac{M^2}{M^2 - 1}\right) \tag{5}$$

Also, by keeping only the lowest order terms in θ, we find, from $\phi = \theta_W - \theta$:

$$\frac{M^2 - 1}{M^2} - \cos^2\phi = 2\sqrt{1 - \cos^2\phi}\,\cos\phi \times \left\{\theta\left[\frac{(\gamma - 3)M^2 + 4}{4(M^2 - 1)}\right]\right\} \tag{6}$$

This equation is solved by iterating, the first approximation being $\cos\phi = \sqrt{(M^2 - 1)}/M$; the next approximation is:

$$\cos\phi = \frac{\sqrt{M^2 - 1}}{M}\left\{1 - \frac{\theta}{2}\,\frac{\sqrt{M^2 - 1}\,[(\gamma - 3)M^2 + 4]}{2(M^2 - 1)^2}\right\} \tag{7}$$

Substituting into Eq. (4) gives:

$$\frac{\Delta P}{q} = \theta\left(\frac{2}{\sqrt{M^2 - 1}}\right) + \theta^2\left[\frac{M^4(\gamma + 1) - 4M^2 + 4}{2(M^2 - 1)^2}\right] \tag{8}$$

which is the Busemann relation.

If, on the other hand, we consider shock waves, we have the well-known relations:

$$\Delta P = \rho_0 W u, \quad W = \frac{\gamma + 1}{4}\,u + \sqrt{\frac{(\gamma + 1)^2}{4}\,u^2 + a_0^2}$$

To the second order we therefore get, as above:

$$\frac{\Delta P}{q} = \frac{2\theta}{M\cos\phi} + \frac{\gamma + 1}{2}\left(\frac{\theta^2}{\cos^2\phi}\right)$$

and for the shock angle:

$$\sin\theta_W = (1/M) + [(\gamma + 1)/4]\,(\theta/\cos\phi)$$

and hence Eq. (8) follows again.

The above method can be extended to unsteady airfoil problems, but the work becomes virtually intractable unless we assume that θ_W is constant during the motion.

References

[1] Laitone, E. V., *Exact and Approximate Solutions of Two-Dimensional Oblique Shock Flow*, Journal of Aeronautical Sciences, Vol. 14, No. 1, pp. 25–42, January, 1947. (Note that the coefficient of θ^2 requires correction in this paper.

[2] Taylor, G. I., British Aeronautical Research Committee, Reports and Memoranda, No. 1467, 1932.

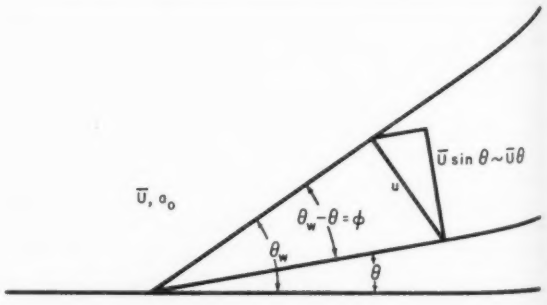

FIG. 1 SKETCH OF FLOW FIELD

On Simple Plan Forms in Supersonic Flow

John W. Miles
Department of Engineering, University of California at Los Angeles
September 26, 1949

IN A RECENT ITEM[1] in this column, we pointed out that the problem of nonsteady motion of a "wide" delta wing could be reduced to an equivalent two-dimensional problem. For the calculation of the force or moment coefficients, this reduction establishes the validity of strip theory (i.e., the calculation of the flow at any spanwise station as if it were two-dimensional).

The purpose of this note is to point out that strip theory is valid for the calculation of the forces and moments on any simple plan form with a straight trailing edge in either steady or unsteady flow. (A simple plan form is one for which the component of free-stream velocity normal to the boundary of the plan form in its own plane is everywhere supersonic, so that the zone of action of the plan form has no region in common with its zone of influence.) This follows directly from the result for the wide delta wing, for we may circumscribe a wide delta about any simple plan form and regard the prescribed normal velocity as null outside of the simple plan form. The restriction to a straight (transverse to the flow) trailing edge follows from the fact that the assumption of zero normal velocity aft of the trailing edge does not meet the physical requirement of zero pressure jump across the wake. Our result may be stated mathematically as follows: If $c_{p_3}(x,y,t)$ is the true three-dimensional pressure coefficient for a simple plan form with a straight trailing edge, if $f(x)$ is an arbitrary (but finite) weighting function, and if S is the plan form, we have:

$$\iint_S (a + by)f(x)\, c_{p_3}(x,y,t)dx\, dy = \iint_S (a + by)f(x)\, c_{p_2}(x,y,t)dx\, dy \quad (1)$$

where $c_{p_2}(x,y,t)$ is calculated on the assumption that the flow at any section is two-dimensional. In the case of steady flow, this result may also be proved with the aid of the reversed flow theorem for simple plan forms.[2] It is important to note that Eq. (1) is valid only for a (at most) linear, spanwise (y direction) weighting function.

References

[1] Miles, J. W., *On Non-Steady Motion of Delta Wings*, Readers' Forum, Journal of Aeronautical Science, Vol. 16, No. 9, pp. 568–569, September, 1949.
[2] Hayes, W. D., *Linearized Supersonic Flow*, North American Aviation, Inc. Report AL-222, June, 1947.

Application of Simultaneous Equation Machines to Aircraft Structure and Flutter Problems

(*Continued from page 113*)

Conclusions

equation machine varies as n^2, while mathematical methods vary as n^3.

No check was made against I.B.M. solutions, since the problems were so small that undue preparation time would have been involved and the comparison, consequently, unfair to the digital machine. It is believed, however, that, because of the flexibility of the exploration of the problems and the intimate way the machine may be integrated into an engineering department, simultaneous equation analogy computers have a place alongside of the digital machines. Digital machines are useful in preparing the data for insertion in the equation machine.

The problems presented here are only an example of what may be done with equation machines, and many problems, some undoubtedly not yet envisaged, may be solved with useful results.

The accuracy is adequate for engineering practice, and the accuracy improvement methods described not only provide a means of improving it if required but also an independent check of the calculations.

A machine with a capacity of between 10 and 20 real equations, equipped with provisions for complex λ's, is recommended as being the proper size for an aircraft engineering department.

Reference

[1] Goldberg, Edwin A., and Brown, George W., *An Electronic Simultaneous Equation Solver*, Journal of Applied Physics, Vol. 19, pp. 339–345, April, 1948.

Erratum

The Readers' Forum item "Supersonic Airfoils Simplified" (JOURNAL, Vol. 17, No. 1, p. 61, January, 1950) was incorrectly attributed to James E. Broadwell. The correct author is **Ralph H. Upson**, *Professor of Aeronautical Engineering, University of Minnesota, Minneapolis, Minn.*

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